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ABSTRACT

The validity and the reliability of a technique for identifying and studying presentation variables in mathematics texts were investigated in this study. A category system for classifying messages in mathematics texts in terms of mathematical content and processes and in terms of mode of representation, procedures for applying this system to texts, and a method for analyzing the information collected through these techniques were developed. Sections of twelve textbooks ranging from grade levels four through twelve as well as six pairs of contrived passages were analyzed using these methods. Results showed that this type of analysis can be used to describe the nature of the presentation in mathematics text passages, that through these methods data are provided on variables which statistically differentiate among textbooks, and that there were no significant differences between groups of raters using this technique to analyze the textbooks. (Author/DT)

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A TECHNIQUE FOR STUDYING THE ORGANIZATION OF MATHEMATICS TEXT MATERIALS

Robert B. Kane and Alan W. Holz

Purdue Research Foundation
West Lafayette, Indiana 47907

November 1972

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ABSTRACT

In this research a technique for identifying and studying presentation variables in mathematics text was developed and investigated. These presentation variables concern the manner in which mathematics text is communicated in printed form. From a developmental point of view the purpose of the research was to develop a technique of studying presentation variables in mathematics text. From an empirical point of view the purpose of the research was to investigate the validity and reliability of the technique.

The developed technique has three basic components. The first component consists of a two dimensional system of categories, called the CMMT category system, for classifying messages in mathematics text. Dimension 1 of the system classifies messages in mathematics text in terms of mathematical content and processes. Dimension 2 of the system classifies messages in terms of mode of representation. The second component of the developed technique concerns procedures for applying the category system to classify messages in mathematics text. The third component, called CMMT analysis, is a method of analyzing the information collected by applying the CMMT categories to mathematics passages. CMMT analysis utilizes a computer program to provide information about an analyzed passage in the form of a sequential list of messages in the passage, interaction matrices showing the relationships among types of messages, and various proportions reflecting the frequency with which types of messages appear. The union of the three components is called the CMMT technique.

The purpose of the validity study was to investigate the validity of the CMMT technique as a means of studying presentation variables in mathematics text. Validity was studied in a descriptive and statistical manner. Sections of twelve textbooks ranging from grade levels four through twelve as well as six pairs of contrived passages were submitted to CMMT analysis. The results of the validity study indicate that (1) CMMT analysis can be used to describe the nature of the presentation in mathematics text passages, (2) passages sampled at random from textbook sections tend to correlate highly with the entire sections from which they are sampled, (3) passages sampled from the same textbook tend to correlate more highly than passages sampled from different textbooks, and (4) CMMT analysis provides data on variables which statistically differentiate among textbooks.

The purpose of the reliability study was to determine within- and between-rater reliability estimates for subjects using the CMMT technique. Three groups of subjects consisting of mathematics education specialists, secondary mathematics teachers, and student teachers of secondary mathematics were trained in the use of the CMMT technique. On two occasions these subjects coded six criterion passages. The results of the reliability study were (1) there were no statistical differences between groups of raters, (2) some passages were statistically more difficult to code than others, (3) between-rater reliability coefficients for dimension 1 categories ranged from approximately .50 to .80 while for dimension 2 they ranged from about .75 to 1.00, (4) within-rater reliability coefficients for dimension 1 categories ranged from about .60 to .90 while for dimension 2 they ranged from about .75 to 1.00, and (5) investigator reliability measured in terms of the codings of subjects averaged .77 for dimension 1 categories and .99 for dimension 2 categories.

The conclusion which may be drawn from the validity and reliability studies is that the CMMT technique has potential for becoming a useful technique for investigating presentation variables in mathematics text. Future research will have to determine if this potential will be realized.

CHAPTER I

INTRODUCTION

Need for the Study

A major component of mathematics instruction is necessarily the instructional materials used. Usually these materials are in the form of printed text. The past fifteen years has seen the production of vast amounts of various mathematical text materials produced by publicly supported writing groups as well as by many independent authors and publishing firms.

Aside from the actual mathematical subject matter, which is essentially common to all recent mathematics textbooks, different writing groups and authors hold differing views on how mathematics should be presented. Advertisements for new textbooks would have the reader believe that each set of mathematics materials offers a unique (and superior) approach to the presentation of mathematics in printed form. While the advertising claims no doubt overemphasize the point, casual comparisons of modern mathematics textbooks reveal that while content seems indeed to be similar, approaches to presenting this content appear varied.

Unfortunately, research concerned with the investigation of presentation variables in mathematics text is practically non-existent. In fact, it is very difficult to even establish what constitutes a given approach to presenting mathematics text, much less to compare various approaches. There have been great numbers of descriptive and comparative studies of "modern" and "traditional" approaches to mathematics instruction and most of these studies include written materials. However, these studies typically confound presentation variables with content and teacher variables making the results, when there are any, uninterpretable in terms of presentation approaches.

There has been some research relating more specifically to written materials than the broad descriptive and comparative studies referred to above. Research has been conducted in the sequencing of mathematical instructional materials and work is continuing in this area (Heimer, 1969). Others have studied the relationship between reading ability and mathematics text. Progress has been made toward adapting ordinary English readability techniques for use on the language of mathematics (Kane, 1970).

Research which effectively studies printed mathematics materials is indeed sparse. Other than in the areas of sequence theory and readability there appears to be no research designed to systematically study mathematics text. Additional significant variables in mathematics text need to be identified and methodologies for investigating these variables need to be developed.

Introduction to the CMMT Technique

In this research a technique for identifying and studying presentation variables in mathematics text was developed. These presentation variables, which concern the manner in which authors attempt to communicate mathematics in printed form, were investigated in terms of various content and representational structures in mathematics text.

The structure of any object may be defined by its elements and by the interrelationships among these elements. These interrelationships concern the sequencing or ordering of the elements and the frequency with which the various elements occur. When considering the presentation structure of mathematics text, the elements may be thought of as messages. The interrelationships among the messages determine the organization of the material.

A mathematics text passage can be thought of as consisting of a series of messages. In terms of formal mathematical content, the basic message types can be conceived of as undefined terms, definitions, axioms, and theorems. Definitions require examples and theorems require proofs. For mathematics to be meaningful, applications are required. To help the student learn the mathematics, exercises and problems are usually provided. In terms of representation of the content, the basic messages can be conceived of as consisting of words, special mathematical symbols, or various types of illustrations.

These considerations of mathematical content and mode of representation along with extensive examination of existing mathematics textbooks led to the development of a two dimensional category system for classifying messages in mathematics text. This system of categories is given next.

Classification of Messages in Mathematics Text
(CMMT Categories)

Dimension 1: Content Mode

	0. Blank Space
Reception (Exposition)	1. Definition (Meaning of words and symbols.)
	2. Generalizations (Rules, axioms, theorems, formulas, etc.)
	3. Specific Explanation (Concrete examples, specific discussion.)
	4. General Explanation (Proofs, general discussion.)
Messages requiring responses. (Exercises, problems, etc.)	5. Procedural Instruction (Directions.)
	6. Developing Content (Questions in exposition, developmental activities, guided discovery exercises, etc.)
	7. Understanding Developed Content (Exercises involving routine computation, practice, identification, etc.)
	8. Applying Developed Content (Real world problems, applications of generalizations in concrete situations, etc.)
	9. Analyzing and Synthesizing Developed Content (Proving propositions, finding new relationships, unguided discovery, etc.)
	10. Other Material (Headings, non-mathematical materials, etc.)

Dimension 2: Representation Mode

	0. Blank Space
Written	1. Words
	2. Mathematical Symbols
Illustrations	3. Representations of Abstract Ideas (Venn diagrams, geometric diagrams, mapping pictures, etc.)
	4. Graphs (Number lines, coordinate graphs, bar graphs, etc.)
	5. Representations of Physical Objects of Situations (Plans, maps, cross sectional drawings, photographs, etc.)
	6. Non-mathematical Illustrations (Motivational photographs, cartoons, etc.)
	7. Combinations of Illustrations with Written Text (Flow charts, mathematical tables, tree diagrams, etc.)

The CMMT category system is applied to mathematics text passages in much the same way that many other observation scales are applied to the phenomena they measure. The basic unit of measure used is one-fourth of a line of print. An area unit is used so that quantitative aspects of the organization of passages can be described in terms of the space devoted to the various types of messages. To apply the category system, a page of text is partitioned into sections conforming to the format of the page. A category number for each dimension of the CMMT system is recorded for each section of this partition.

When classification of all messages in a passage is completed the information may be analyzed in ways which reflect both sequential and quantitative aspects of the organization of the passage. Sequential aspects are represented by making an ordered list of the classifications following the natural flow of the printed material. Matrices, similar to those used in interaction analysis (Flanders, 1970, pp 77 - 86) are used to analyze the nature of the interactions among the CMMT categories in each dimension. Quantitative aspects of a passage are described by determining the proportions of messages of various types and logical combinations of types. All of the descriptive information for a given passage is called the CMMT analysis for that passage. A computer program is used to derive the CMMT analysis of a passage from a rater's codings.

Thorough descriptions of the CMMT categories and specific procedural rules are used by raters applying the CMMT category system to mathematics text passages. The procedural rules describe general procedures for raters to follow and give specific decision rules for dealing with such problems as the classification of units containing more than one type of message and what to do when in doubt about the classification of a message. A method of estimating rater reliability adapted from interaction analysis (Flanders, 1960) is used to estimate the reliability of CMMT raters.

In summary, this study concerned the development of a technique for studying presentation variables in mathematics text. The basic components of the technique are the CMMT *category system, the procedures for using the CMMT categories,* and CMMT analysis. The union of these components is called the *CMMT technique*.

Overview of the Study

This study had purposes which were both developmental and empirically oriented. From a developmental point of view, the purpose of the study was to develop and refine the CMMT technique. From an empirical research point of view, the purpose of the study was to investigate the reliability and validity of the CMMT technique as a measurement instrument.

The development of the CMMT technique was an evolutionary process. The germinal idea was developed from classroom interaction analysis studied by Flanders (1970), Amidon (1967) and others. The idea was to use a technique similar to interaction analysis to study mathematics text. The first step in the development of such a technique was to examine existing mathematics textbooks and to begin forming a classification system. Successive tentative category systems were developed, tried out on mathematics textbooks, submitted to critical discussion, and revised. As the category system was being developed a theoretical rationale for the choice of the categories also began to evolve. Thus, the final form of the CMMT category system was based on both practical experience and theoretical considerations.

Procedures for using the system to code text materials were also developed. Procedural matters which were dealt with included the choice of the unit of observation, decision rules, sampling procedure, and the scope or range of application. Simultaneous with the development of the CMMT category system and procedures, the method of CMMT analysis and a method of estimating rater reliability were developed. Computer programs were written for deriving the CMMT analysis of a passage from a rater's codings and for determining rater reliability estimates. Chapter 3 is devoted to a detailed discussion of the development of the CMMT technique.

The specific purposes of the empirical part of the study were:

1. To investigate the validity of the CMMT technique as a means of studying the organization of mathematics text.
2. To investigate the reliability of raters trained in the use of the CMMT technique.

To carry out these purposes, twelve textbooks ranging from grade four through grade twelve were studied. A total of 99 passages from the twelve textbooks plus twelve contrived passages were submitted to CMMT analysis in the validity portion of the study. Certain focus passages were used to demonstrate that CMMT analysis can be used to describe the organization of mathematics text. The contrived passages, which were written in pairs following opposing organizational plans, were used to show that CMMT analysis reflected these plans. Successive correlations were made between entire sections of the textbooks and increasing samples of passages chosen randomly from these sections using data from the CMMT analyses. These correlations were used to determine what constituted an adequate sample of passages from the textbooks. Passages within textbooks were also correlated using data from the CMMT analyses. These correlations were compared to correlations between passages from different textbooks to determine if passages within textbooks correlated more highly than passages between textbooks. Finally, an analysis of variance model was used to identify differences between the twelve textbooks on a number of CMMT organizational variables.

Three groups of subjects including university seniors studying to become secondary mathematics teachers, practicing secondary mathematics teachers, and professional mathematics education experts were used to study rater reliability. The subjects in each group studied a training booklet to learn the CMMT technique and then rated six criterion passages on two occasions separated by a minimum four week period of time. Repeated measures analyses of variance were performed to determine if within- and between-rater differences existed between the groups of subjects and to determine if differences existed in the rating difficulty of the criterion passages. Within- and between-rater reliability coefficients were determined for the subjects in each group who obtained proficiency in the application of the CMMT technique. Chapters 4 and 5 are devoted to a discussion of the reliability and validity studies.

Significance of the Study

The CMMT technique has import for both practical application and research. Since CMMT analysis provides a means of describing the presentation of mathematics text in a specific concrete manner, it could be used to provide valuable information about existing textbook materials. Publishers may wish to use CMMT analysis to help describe the nature of their textbooks. Authors and editors of mathematics textbooks may wish to use CMMT analysis to assist them in the writing and revision of text material. Textbook selection committees may wish to use CMMT analysis to help them gain more information about the textbooks they are considering. CMMT analysis could be utilized in mathematics teacher education programs as an aid in the study of mathematics materials. Thus, CMMT analysis could become a valuable practical tool for helping publishers, authors, textbook selection committees, and students of mathematics education gather information about mathematics instructional materials.

From a research point of view, the CMMT technique offers a way of manipulating and controlling variables in comparative studies of written mathematics text materials. Researchers could contrive materials which systematically vary on a number of CMMT organizational variables. The relationships between these variables and learner outcomes could then be investigated. Since the CMMT technique can be applied to any mathematics text materials, it is possible that results of completed comparative studies could be more meaningfully interpreted by submitting the materials used in these studies to CMMT analysis. Researchers may wish to use CMMT analysis to search for organizational differences among existing mathematics textbooks. Thus, CMMT analysis could prove useful in research for both identifying and systematically studying organizational variables in mathematics text.

CHAPTER II

BACKGROUND OF THE STUDY

The first thrust of the revolution that swept mathematics education in the post-Sputnik era had to do with changing the content of school mathematics. A comparison of the tables of contents of textbooks used today with those used in the 1950's reveals the success of this part of the revolution.

A second thrust of the school mathematics revolution dealt with how mathematics should be presented to students. Romberg (1969) states that while there is now considerable agreement about what mathematics should be taught there is not agreement about how it should be taught. Clearly, how mathematics should be taught is closely related to how it should be presented to students in written form. This chapter discusses presentation approaches employed by authors of mathematics textbooks, research on mathematics text which has been carried out, and research methodologies related to those used in this study.

Approaches to Presenting Mathematics in Text Form

Different writing groups and authors hold differing views on how mathematics should be presented in written form. Beberman (1958) described the presentation approach utilized by the University of Illinois Committee on School Mathematics (UICSM). This approach which has come to be known loosely as "guided discovery" used problems and exercises to develop concepts and skills. The UICSM approach was based on the belief that the student will come to understand mathematics if he plays an active part in developing mathematical ideas and procedures. According to Beberman, UICSM materials stressed precision in the use of mathematical language with the final verbalization of concepts being delayed until after the student has worked with the concepts and come to understand them.

Wooten (1965) described the more traditional expository presentation approach taken in materials written by the School Mathematics Study Group (SMSG). Typically SMSG materials consisted of an exposition of a mathematical concept followed by exercises and problems over the developed content. Interesting applications and extensions of the developed content were included periodically as problems; however, subsequent content development was not usually dependent on these problems. Like UICSM materials, SMSG materials stressed the precise use of mathematical language but no attempt was made to provide extensive experience with concepts before verbalization.

The differences of presentation exemplified by UICSM and SMSG materials are typical of the types of differences in presentation which can be found in other mathematics text materials. An examination of existing mathematics textbooks reveals a number of apparent presentation variables in mathematics text. These apparent variables include quantitative variables such as the amounts of verbal material, symbolic material, illustrative material, blank space, expository material, exercises, problems, etc. They also include sequential variables such as the degree of integration of exercises with exposition, the placement of illustrations, the order of presentation of examples and generalizations, etc. Finally, these variables include certain qualitative variables such as the levels of abstraction, concreteness, generality, application, etc.

The advertisements of textbook publishers also indicate the existence of a number of approaches to the presentation of mathematics in text form. The advertisements claim that various books utilize the discovery approach, the manipulative approach, the structural approach, the axiomatic approach, the informal approach, the inductive approach, the workbook approach, the application approach, the spiral approach, the programed text approach, etc. While it is not always clear exactly what all of these terms mean, the advertisements support the contention that there are a variety of approaches to presenting mathematics in printed form.

In summary, there are apparent identifiable presentation variables in mathematics text. Some of these variables have been described by authors such as Beberman and Wooten. Others can be inferred from the examination of existing textbooks and from advertisements for new mathematics textbooks. Unfortunately, research relating specifically to presentation variables in mathematics text is practically nonexistent. There has been some research relating to other specific variables in mathematics text and to mathematics text in general. This research is discussed in the following sections.

Descriptive and Comparative Studies of Mathematics Text

Romberg (1969) reviewing current research in mathematics education reported little research specifically concerned with mathematics text. He reported a number of descriptive studies related to the effectiveness of instructional programs and many of these studies included written materials. However, the main emphasis of these studies appeared to concern the mathematical content—i.e., could a certain topic be taught to a given group of students. No control or comparison groups were present in these studies and instructional variables other than written text were present to confound any results. Hence, these descriptive studies only remotely relate to the study of mathematics text and no generalizations about variables in mathematics text can be made from them.

Romberg (1969) also reported that numerous studies have been conducted comparing "modern" and "conventional" mathematics instruction and most of these studies included written materials. However, he found it difficult to make any generalizations about the effectiveness of written materials from these studies. Most of the studies were small "one shot" affairs in which no significant differences were found. When differences were found it was usually not clear how treatments actually differed and confounding variables such as teacher influence and between student interactions were usually present. Thus, the results of these comparative studies were uninterpretable in terms of the learning materials used.

Kieren (1969) reviewed the research in discovery learning in mathematics and found that published research in the area was large in volume and generally poor in quality. There have been a great number of studies comparing "discovery" and "traditional" approaches to mathematics teaching which have utilized written materials. Kieren noted, however, that each researcher seemed to have a different conception of what constituted a "discovery" and a "traditional" treatment. In addition to this lack of operational definition of treatments, he suggested that discovery research generally suffers from a bias of having the development effort expanded on the discovery treatment. Hence, it is very difficult to make many conclusions about mathematics text from the research reported in Kieren's review.

By far the most comprehensive study comparing mathematical text materials is the National Longitudinal Study of Mathematical Abilities carried out by the research group of the School Mathematics Study Group (Wilson, et al., 1969).

In that study the relative effects of a number of mathematics textbook series were studied over a period of several years. A vast amount of data concerning achievement and attitude was collected and submitted to intense and sophisticated analysis. While the study avoided many of the problems inherent in smaller comparative studies the results do not shed much light on the question of what constitute effective approaches to presenting mathematics in written form. Apparently, all that can be said about written materials from the results of the study is that book A appeared to be superior to book B in some aspect of student achievement or attitude. It is impossible to identify what variables made the approach of book A superior to that of book B.

Thus, in the past there have been great numbers of descriptive and comparative studies involving mathematics text. These studies have ranged from small "one shot" studies to the extensive National Longitudinal Study of Mathematical Abilities. A search of research since the time of Romberg's and Kieren's reviews in 1969 reveals that little has changed. Descriptive and comparative studies involving mathematics text continue to be done. The recent studies appear to suffer from the same deficiencies described by Romberg and Kieren and the differences, when there are any, do little to identify specific meaningful variables in mathematics text.

Both Romberg and Kieren in their reviews infer a need for long range systematic research of mathematics text.

Important variables in mathematics text need to be identified and methodologies for studying them need to be developed. Descriptive and comparative studies involving mathematics text have done little to meet this need. Fortunately, there has been some research relating more specifically to mathematics text than these broad types of studies. This research is discussed in the next section.

Systematic Research of Mathematics Text

Two areas where some systematic research of mathematics text has been carried out are the areas of sequence theory and the readability of the language of mathematics. These areas of research will now be discussed in turn.

Sequence Theory

Heimer (1969) described the research and development efforts toward producing an adequate sequence theory in mathematics instruction. He defined instructional sequence to mean the order in which the learner interacts with the units of content. Heimer claimed that every effort to construct mathematical materials demands decisions about structuring the content and designing and ordering instructional tasks. Thus, sequence theory has important implications for the study of variables in mathematics text.

A major theoretical formulation of sequence theory which has implications for the sequencing of mathematics text concerns Gagne's learning hierarchies. According to

Gagne (1968) a subject may be organized into hierarchical structure with a terminal task at the top and prerequisite tasks below. To learn the subject a learner starts at the bottom of the hierarchy and works his way up through the prerequisite tasks to reach the terminal task at the top. This theoretical formulation has been particularly appealing for the design of sequences in mathematics text since mathematics has a logical structure which is hierarchical in nature.

Heimer (1969) claimed that a critical analysis of the nature and role of learning hierarchies gives rise to a number of issues which need exploring. These include:

1. How is a learning hierarchy constructed?
2. How is the validity of a learning hierarchy to be determined?
3. What is the relationship between an hypothesized learning hierarchy and the associated presentation sequence for instruction?
4. What is the connection between the logical structure of the content and the associated learning hierarchy?

Heimer in his review indicates that a number of studies (e.g., Gagne & Paradise, 1961; Gagne, 1962; Gagne et al., 1962; Merrill, 1965; and Briggs, 1968) have explored these questions and he called for continued systematic research in the area. Such studies have obvious implications for the design of sequences in mathematics text.

The position of Ausubel (1963) on learning hierarchies and sequencing is not unlike Gagne's. Ausubel claims that most tasks can be analyzed into a hierarchy of learning units. He stated that the sequential organization of subject matter can be useful with each new increment of knowledge serving as an anchor for subsequent learning. Ausubel's conceptualization of advance organizers, which appear to consist of general non-technical overviews or outlines of the content to be learned, also has implications to the construction of sequences in mathematics text. Several studies by Ausubel and others (e.g., Merrill & Stolurow, 1965; Ausubel, Robbins & Blake, 1957; Woodward, 1966; and Scandura & Wells, 1967) concerning advance organizers have been conducted, but the results are inconclusive. A major problem with these studies seems to be that no generally accepted operational definition of advance organizers has been established.

Suppes (1967) has been another contributor to the theory of sequencing in mathematics instruction. Suppes subscribes to the idea of the importance of content structure in the study of learning sequences. He has theorized about the connection that exists between the psychological processes of acquisition and the logical structure of mathematical concepts. He has made only a few tentative hypotheses regarding this and systematic long-range research is needed in the area. The implications such research would have for the design of mathematics text are clearly important.

There have been many studies concerning sequence theory other than those closely related to the theories of Gagne, Ausubel, and Suppes. Some researchers (e.g., Roe, Case, & Roe, 1962; Roe, 1962; Payne, Krathwohl & Gordon, 1967; and Pyatte, 1969) have studied the effects of "logically ordered" versus "scrambled" sequences in programed text, but the overall results have been inconclusive. Heimer (1969) indicated that the effect of scrambling may be dependent of the size of the learning unit which was scrambled, on the logical interrelatedness of the content being presented, and on the validity of the order of the "logical" sequence which was scrambled. Heimer questioned the purpose of studies of scrambled order sequences and inferred that the results of such studies add little to the knowledge of how to construct effective sequences.

A number of studies reported in Heimer's review concerned the evaluation and/or contrasting of sequence strategies in mathematics materials. These studies involved the tree-graph structuring of content into concepts and subconcepts (Newton and Hickey, 1965), the effect of structurally related classes of previous learning on problem solving (Scandura; 1966a, 1966b, 1966c), multiple concept versus single concept sequences (Short and Haughey, 1967), and variables of repetition and spaced review (Reynolds et al., 1964). Hickey and Newton (1964) also set forth a set of eight hypotheses for the study of structure and sequence.

In concluding his review of the research in sequence theory, Heimer stated that knowledge about the construction of efficient instructional sequences in mathematics materials was desperately sparse. He indicated that while a number of researchers are working in the area there is a need for systematic long-range programs of research of the sequencing of instructional materials in mathematics.

The research in sequence theory briefly described here is only tangentially related to the type of presentation variables in mathematics text investigated in this study. Hence, the research in sequence theory was not reviewed in depth. An area of research more closely related to this study is the area of the readability of mathematical language discussed in the next section.

Readability Studies

Readability formulas have been widely used to assess the ease with which written material can be comprehended by readers. A readability formula is a prediction equation. The predictor variables typically include such variables as average sentence length, the number of difficult words according to some standardized vocabulary list, the number of personal pronouns, and others. These predictor variables depend only on the reading materials and not on the reader. The criterion measure, on which a readability formula is validated, is some measure of the comprehensibility of the material such as readers' comprehension test scores, expert judgement, or other validated readability formulas.

Thus, a readability formula can be applied to written text independent of the reader to assess the readability of the material. Care should be taken, however, when applying readability formulas to written material. Any measurement instrument is valid for a given purpose, for specific subjects, and specific content. Therefore, a given readability formula should only be applied to measure the readability of material similar to the material which was used in its validation and for subjects comparable to the subjects on which it was validated.

Most readability formulas were validated using materials of a general reading nature. Kane (1968, 1970) described some of the differences between ordinary English and mathematical English. For example, he pointed out that mathematical English has a specialized vocabulary and contains special symbols not found in ordinary English. He discussed the inappropriateness of applying existing readability formulas to mathematical writing. Chall (1958, p. 125) noted that it is questionable whether existing readability formulas are valid for estimating reading difficulty of mathematics textbooks.

Nevertheless, a number of researchers (Heddens & Smith, 1964a, 1964b; Wiegand, 1967; Smith, 1969; Cramer & Dorsey, 1969) have used standard readability formulas in an attempt to measure the readability of mathematics textbooks. In these studies mathematical words were usually classified as difficult words and mathematical symbols were ignored in applying the formulas. For reasons described above the results of these studies are highly suspect.

A few researchers have made progress toward adapting regular readability techniques to the language of mathematics. The cloze procedure has been used by researchers as a measure of the readability of ordinary English. Cloze tests are constructed by deleting words or symbols from passages and replacing them with blanks which subjects attempt to fill-in with the correct word. Hater (1969) validated the cloze procedure as a measure of reading comprehension on the language of mathematics at grade levels 7 through 10.

Most readability formulas utilize some measure of vocabulary difficulty. Because of the specialized nature of mathematical vocabulary and mathematical symbolism, word familiarity lists constructed for ordinary English are inadequate for measuring the familiarity of mathematical language. Byrne (1970) established a measure of the familiarity of mathematical terms and symbols to seventh and eighth grade students in the United States.

Building on Hater's and Byrne's research Kane, Hater, and Byrne (1970) constructed a readability formula for the language of mathematics. This formula predicts mean cloze scores for seventh through tenth grade students on mathematics passages. The predictor variables in this formula were the number of mathematics words not on the list of mathematics words judged known by 80 percent of the seventh and eighth graders in the United States and the number of different words with three or more syllables. These variables accounted for 60 percent of the variation in mean cloze scores for the criterion passages. These researchers are continuing work toward the construction of a second formula based on a broader sample of materials than the initial formula.

Kulm (1971) studied the readability of elementary algebra textbooks. He used multiple regression analysis with criterion measures of mean cloze scores to search for predictors of readability. He found that the percentage of mathematical symbols, the percentage of mathematical vocabulary words, average sentence length, and the percentage of questions were the best four predictors accounting for approximately 40 percent of the variation.

While the research in the readability of the language of mathematics is not directly related to the technique of studying presentation variables in mathematics text developed in this study, one major aspect of the readability research is similar. Readability formulas provide a means of obtaining specific information about mathematics text. This study represents an extension of readability studies in the sense that its purpose was to develop a technique for obtaining other useful information about mathematics text. On the other hand, the methodology of constructing prediction equations is not employed in this study. Methodological techniques and other research directly utilized in this study are described next.

Methodological Techniques and Other Related Research

The germinal idea for the technique of investigating presentation variables in mathematics text which was developed in this study came from the systems of classroom interaction analysis developed by Flanders (1970), Amidon (1967) and others. These systems of interaction analysis utilize sets of categories for classifying types of student and teacher classroom behaviors. The classifications of the behavior in classrooms are then analyzed to describe the nature of the interaction which has taken place.

An observer utilizing an interaction analysis system classifies the type of behavior occurring in a classroom at time intervals of say every three seconds. The sequence of these classifications is then listed to describe the sequence of behaviors which occurred in the classroom. The percentage of observations in each category or logical combinations of categories are computed to describe the amount of time devoted to various behaviors. Interactions which exist between the classified behaviors are analyzed with the use of a matrix. For an n category system, an $n \times n$ matrix is constructed with the ij^{th} entry representing the frequency with which category i was followed by category j in the observer's sequence. Various interactions between the classified behaviors can be described in terms of the entries in certain regions of the matrix.

The technique of interaction analysis has led to a great deal of fruitful research in analyzing teaching behavior. Application of interaction analysis has been made to problems of improving teacher behavior and to the training of teachers. While interaction analysis research and its applications are interesting, it is the technique of interaction analysis which is important to the study presented here. The idea of this study was to develop a system of categories for classifying material in mathematics text and to use a technique similar to interaction analysis to analyze the interrelationships among classifications.

The first major problem to be faced was the construction of a category system for classifying the material in mathematics text. As a theoretical starting point three established classification systems were drawn upon. Bloom (1956) presented a system for classifying educational objectives. This hierarchial system is summarized below.

1. Knowledge - involves the recall of specifics, methods, or processes.
2. Comprehension - involves the understanding and use of material without relating it to other material or seeing its full implications.

3. Application - involves the use of abstractions in particular and concrete situations.
4. Analysis - involves the breakdown of material into its constituent elements so relations between ideas are made explicit.
5. Synthesis - involves the putting together of elements so as to form a whole.
6. Evaluation - involves judgments about the value of material.

The National Longitudinal Study of Mathematical Abilities (Wilson, et al., 1969, pp. 39-40) developed a system not unlike the Bloom system to classify materials for testing mathematical abilities. These categories are summarized below.

1. Computation - straight forward manipulation according to rules.
2. Comprehension - emphasis on demonstrating understanding of concepts and relationships.
3. Application - requiring recall of knowledge, selection of operations, and performance of operations.
4. Analysis - requiring a non-routine application of concepts.

Finally, Bruner (1967) described a system of representation which had implications for the category system developed in this study. He theorized that the representation of material may be symbolic, iconic, or enactive. Symbolic representation is exemplified by languages and systems of mathematical symbols. Iconic representation concerns pictures, illustrations, and physical models. Enactive representation involves the manipulation of real objects.

Summary

There are apparent variables in the presentation of mathematics text. These variables have been described by mathematics textbook authors and can be inferred from observations of textbooks. Research related to mathematics text consists of descriptive studies, comparative studies, sequencing studies, and readability studies. Little is known from the research about presentation variables in mathematics text.

In this study the technique of classroom interaction analysis served as a model for the development of a technique for studying presentation variables in mathematics text. Classification systems of Bloom, the National Longitudinal Study, and Bruner served as a theoretical starting point for establishing a system of categories for classifying material in mathematics text. The category system and technique which were developed are described in the following chapter.

CHAPTER III

DEVELOPMENT OF THE CMMT TECHNIQUE

One of the main purposes of this study was to develop a technique for identifying and studying presentation variables in mathematics text. These presentation variables concern the manner in which authors attempt to communicate mathematical content in printed form.

Presentation variables in mathematics text were studied in terms of both content and representational structures within mathematics materials. The structure of any object is determined by its elements and by the interrelationships among its elements. In considering presentation structures of mathematics text, the elements are called *messages*. The interrelationships among the messages in mathematics text determine the *organization* of the material.

As a first step in the development of this technique for studying the presentation of mathematics text, a means of identifying the basic elements or messages had to be developed. To accomplish this, a system of categories for classifying messages in mathematics text was devised. The categories in this system define the message types investigated in this study. The classification system is called the CMMT category system. The letters CMMT stand for "Classification of Messages in Mathematics Text."

The second step in the development of this technique was to devise a method of describing and analyzing the interrelationships among the messages in mathematics text. These interrelationships, which determine the organization of mathematics text, concern the ordering or sequencing of the messages and the frequencies with which the messages occur. The method which was developed for analyzing the organization of the messages is called CMMT analysis. The CMMT category system and method of analysis along with the procedures for using the category system are the basic components of the CMMT technique.

The specific developmental objectives of the study were:

1. To develop a system of categories for classifying the messages in mathematics text.
2. To develop complete, usable descriptions of these categories.
3. To develop procedures for using the category system to classify messages in mathematics text.
4. To develop a method of analyzing the organization of mathematics text from the classification system.
5. To develop a method of estimating reliability for raters using the category system.

This chapter is devoted to describing how each of these developmental objectives was attained.

Development of the CMMT Category System

The CMMT category system grew out of considerations which were both theoretical and practical in nature. Theoretical considerations concerned the structure of mathematical systems, cognitive processes used in learning mathematics, and systems of representation of learning material. Practical considerations concerned the nature of existing mathematics textbooks—i.e., the types of things which are possible to classify in mathematics text. The CMMT category system which was included in Chapter 1 is repeated here for convenience. It is followed by a description of its development.

Classification of Messages in Mathematics Text

(CMMT Categories)

Dimension 1: Content Mode

	0. Blank Space	
Messages requiring only reception.	1. Definition (Meanings of words or symbols.) 2. Generalizations (Important rules, axioms, theorems, formulas, etc.) 3. Specific Explanation (Concrete examples and discussion in specific terms.) 4. General Explanation (Proofs of propositions, general discussion, etc.)	Exposition

Messages calling for responses other than reception.	5. Procedural Instructions (Directions.) 6. Developing Content (Questions in exposition, developmental activities, guided discovery exercises, etc. 7. Understanding Developed Content (Exercises involving routine computation, practice, identification, etc.) 8. Applying Developed Content (Real world problems, applications of generalizations in concrete situations, etc.) 9. Analyzing and Synthesizing Developed Content (Proving propositions, finding new relationships in developed content, unguided discovery, etc.	Subclassifications of questions, exercises, problems, activities, etc.
	10. Other Material (Headings, non-mathematical materials, etc.)	

Dimension 2: Representation Mode

	0. Blank Space
Written text.	1. Words 2. Mathematical Symbols
Illustration	3. Representations of Abstract Ideas (Venn diagrams, geometric diagrams, mapping pictures, etc.) 4. Graphs (Number lines, coordinate graphs, bar graphs, etc.) 5. Representations of Physical Objects or Situations (Plans, maps, cross sectional drawings, photographs, etc.) 6. Non-mathematical Illustrations (Motivational photographs, cartoons, etc.)
	7. Combinations of Illustrations with Written Text (Flow charts, mathematical tables, tree diagrams, etc.)

The development of the CMMT category system began with the examination of a number of existing mathematics textbooks. From the beginning the broad division between exposition and exercises or problems was apparent. It was obvious that books differed in the amount of space devoted to exposition versus exercises and problems. From these initial observations the broad categories of messages requiring only reception versus messages calling for responses beyond reception were abstracted.

Another major distinction made clear by the examination of existing textbooks concerned the mode of presentation of mathematics text. Materials appeared to vary with respect to the amounts of words, mathematical symbols, and illustrations which were present. It was clear that there were interactions between these modes of representation and classifications concerning content. Thus, it was decided to construct a two dimensional system of categories with dimension 1 classifying mathematical content and dimension 2 classifying the type of representation used.

As the category system began to evolve, a theoretical rationale for the choice of categories took shape. The subclassifications of exposition in the content mode are based on considerations of the structure of mathematical systems. The elements of the structure of a mathematical system may be thought of as undefined words, definitions, axioms, and theorems. Theorems are statements requiring verifications which are called proofs. Many times general mathematical concepts are explained in terms of specific instances which are called examples. It was decided to classify messages requiring only reception on the part of the learner in terms of these elements of the structure of mathematical systems.

In order for mathematics to become meaningful to a learner, he must interact with it. To help meet this need most textbooks provide exercises and problems. Some textbooks contain other interrogative material woven into the exposition. The problems and questions in mathematics textbooks play various roles in helping students learn mathematics. Some exercises serve to make the learner participate in the development of content. Others aid the learner in understanding concepts of skills. Some problems require the student to extend developed content through application, analysis, and synthesis. Thus, it was decided to classify messages in the content mode requiring responses beyond reception in terms of the cognitive processes required of the learner.

Bloom (1956) provided the hierarchical classification of cognitive processes listed below.

1. Knowledge
2. Comprehension
3. Application
4. Analysis

5. Synthesis
6. Evaluation

The National Longitudinal Study of Mathematical Abilities (NLSMA) utilized a system of classifications similar to the Bloom classifications (Wilson, et al., 1969, pp. 39-40). The NLSMA system is listed next.

1. Computation
2. Comprehension
3. Application
4. Analysis

The subclassifications of problems and exercises used in the content mode of the CMMT category system are similar to the two classification systems given above. The relationship between dimension 1 of the CMMT system and these other two systems is summarized in Figure 1.

Dimension 2 of the CMMT category system deals with how mathematical content is represented. The rationale for the subclassifications of this dimension is derived from Bruner's system of representation (Bruner, 1967, pp. 10-14), from the literature on the readability of the language of mathematics, and from the level of abstraction present in mathematics text illustrations. Brunner theorizes that the representation of material may be symbolic (language), iconic (pictorial), or enactive (manipulative). Although mathematics text by nature is not enactive the symbolic-iconic distinction is an apparent representational variable in mathematics text. This

Structure of a Mathematical System	Classifications		
	CMMT	Bloom	NLSMA
Undefined Words Definitions	1	1 & 2	Not Applicable
Axioms Theorems	2		
Examples	3		
Proofs	4		
Processes Used in Studying Mathematics Text			
Developing Content	6	1 & 2	1 & 2
Understanding Developed Content	7		
Applying Developed Content	8	3	3
Analyzing and Synthesizing Developed Content	9	4 & 5	4

Note: Blank Space (0), Procedural Instructions (6), and Other Materials (10) are essentially non-mathematical in nature and are excluded from the above table. However, these types of messages occur frequently in mathematics textbooks and therefore are included in the CMMT system.

Figure 1

Rational for Dimension 1 CMMT Categories distinction provides the rationale for the broad classifications of written text and illustration.

Readability literature indicates that the distinction between words and mathematical symbols may be an important variable related to the readability of the language of mathematics (Kane, 1970). This distinction serves as the rationale for the subclassification of written text into words and mathematical symbols. The subclassifications of illustration are based on the level of abstraction of the illustration. Categories (3) through (5) of dimension 2 of the CMMT system represent a range in levels of abstraction with (3) representing the most abstract and (5) the least abstract type of illustration.

In summary, the rationale for the choice of the CMMT categories is based on both practical and theoretical considerations. As the system evolved decisions as to what classifications would be theoretically meaningful had to be made in the light of the practicality of using the system to classify messages in existing mathematics text. Several tentative classification systems were developed, tried out on existing materials, submitted to critical discussion, and revised. The classification system which in the end was developed is not the only possible system. Further research will have to determine if it is a system which is usable and meaningful.

Description of the CMMT Categories

As the final form of the CMMT category system began to take shape, tentative descriptions of the categories were written. These descriptions were used in the classification of messages in existing text. Problems in applying these descriptions were noted and on the basis of this experience the descriptions were revised and refined. The resulting complete descriptions follow.

Description of Dimension 1 Categories

0. Blank Space: Any unit which is completely blank.
1. Definition (Meanings of words or symbols.): Messages whose primary purpose is to give the mathematical meanings of words, symbols, or phrases. Any statement of a definition or any description of a mathematical term. May be integrated into the written text or set off as a formal definition.
2. Generalization (Important rules, axioms, theorems, formulas, etc.): Messages whose primary purpose is to give important general mathematical concepts, ideas, or procedures. Any statement of an axiom or theorem. May be integrated into the written text or formally set off as a rule, formula, principle, etc.
3. Specific Explanation (Concrete examples and discussion in specific terms.): Messages whose primary purpose is to give specific or concrete instances of general definitions, axioms, or theorems. Any

discussion or explanation of concrete examples. May be integrated into written text or specifically set off as formal examples.

4. Explanation (Proofs of propositions, general discussion, etc.): Messages whose primary purpose is to explain or discuss general concepts, ideas or procedures. Any proof of a formal proposition. General statements which seek to clarify, justify, show how or why, etc. May be used to introduce, relate, or summarize aspects of a passage. Expository statements with mathematical content which are not clearly classifiable as 1 through 3.

5. Procedural Instructions (Directions.): Procedural directions appearing anywhere in a text passage. May indicate to the reader what he is to do with a set of exercises or problems. May be stated in the form of a question. Does not apply to directive statements containing substantial mathematical information.

   Typical examples:

   a. Solve the following equations.
   b. Complete.
   c. Look at Figure 7.
   d. What are the solution sets for the open sentences below?

6. Developing Content (Questions in exposition, developmental activities, guided discovery exercises, etc.):

Questions, activities, or exercises designed to help develop content. May call for verbal discussion, physical activity, or thought or written responses. May be designed to guide the student to discover some mathematical concept. Exercises or problems whose primary purpose is to develop or present new content. Questions, activities, or exercises integrated into the exposition which are designed to make the reader participate in the development of the content.

7. *Understanding Developed Content* (Exercises involving routine computation, practice, identification, etc.): Exercises which are clearly on the knowledge or comprehension level of Bloom's Taxonomy. May involve lowest level of application where a generalization is used in a situation essentially identical to the situation in which the generalization was presented in the text. The emphasis here is on understanding the content which was presented in the text. May involve the recall of specific information and procedures or the understanding of presented content with only routine alterations of material. May involve the use of material without relating it to other materials or situations or seeing its broader implications.

Typical examples:

a. Exercises of a purely computational nature.

b. Exercises intended to give practice in a skill or procedure.

c. Exercises which are essentially the same as examples given in the text. That is, exercises whose solutions depend on the imitation of examples worked out in the text.

d. Problems which require only a direct routine application of a definition or generalization. Such as finding the circumference of a circle with a given radius. (Where the emphasis is on understanding the generalization rather than on using it to find solutions to problems.)

8. Applying Developed Content (Real world problems, applications of generalizations in concrete situations, etc.): Exercises or problems which are on the application level of Bloom's Taxonomy. The emphasis here is on *using* the content presented in the text in a new or different situation. The new situation may be mathematical or a real life or physical situation. May involve the use of an abstraction in a concrete situation which the student has not seen before. Involves the recall and understanding of information and in addition the utilization of the content in a non-routine manner to find the answer to a problem.

Typical examples:

a. Using general formulas or procedures to solve mathematical problems.

b. Using general mathematical principles developed in the text to find solutions to real life problems. (Word problems.)

c. Applying general definitions, axioms, and theorems in concrete situations or to specific mathematical models.

d. Interpreting concrete examples in terms of general structures.

9. Analyzing and Synthesizing Developed Content (Proving propositions, finding new relationships in developed content, unguided discovery, etc.): Problems which are on the analysis or synthesis level of Bloom's Taxonomy. May involve the breakdown of content into its basic elements so that relationships between ideas are made more explicit.

Typical examples:

a. Problems requiring the proof or disproof of a proposition.

b. Problems requiring the discovery of new relationships between elements of developed content.

10. Other Material (Headings, non-mathematical materials, etc.): Messages not fitting categories 0 through 9. May be headings of sections, exercises, etc. May be

motivational or historical material. Messages which are non-mathematical in character.

Description of Dimension 2 Categories

0. Blank Space: Any unit which is completely blank.
1. Words: Messages which are made up predominately of ordinary English words.
2. Mathematical Symbols: Messages which are made up predominately of mathematical symbols.
3. Representations of Abstract Ideas: May be drawings to illustrate abstract sets, Venn diagrams, mapping pictures, geometric diagrams, etc.
4. Graphs: May be bar graphs, line graphs, circle graphs, etc. Or, may be number lines or various types of coordinate system graphs.
5. Representations of Physical Objects or Situations: Must have some mathematical content. May be drawings or photographs of real objects or living things which illustrate mathematical content. Could be cross sectional diagrams, maps, plans, charts, etc.
6. Non-mathematical Illustrations: Non-mathematical in nature and not illustrating any mathematical content being considered. May be motivational or historical cartoons, photographs, or drawings.
7. Combinations of Illustrations with Written Text: May be some sort of diagram which contains substantial mathematical information in the form of words

or symbols. Includes all mathematical tables, flow charts, tree diagrams, etc.

Procedures for Using the CMMT System

As the definitions of the CMMT categories were being developed a number of procedural matters dealing with how to use the system in classifying messages in mathematics text had to be considered. These procedural matters concerned the unit of observation, the method of subdividing material for classification, coding procedures, specific decision rules to use when deciding between conflicting classifications, general approaches to classifying passages, the scope or range of application of the system, and the sample size to use when analyzing material in existing textbooks.

Unit of Measure

It was decided that an area unit of measure would be used so that the material in a passage could be represented quantitatively in terms of the space devoted to various categories and to logical combinations of categories. From experience rating textbook passages it was learned that, except for the symbol-word classifications, messages as classified by the CMMT system usually took up more area than one-fourth of a line of print. It was also found that units smaller than one-fourth of a line of print tended to make the task of classifying the messages in a passage too tedious. Thus, the area of one-fourth of a line of print on a given page was chosen as the basic unit of measure for that page. A "greatest part rule" was made for classifying units containing more than one type of message.

Partitioning Passages into Messages

Initially it was thought that to code a page of text all that was needed was to arbitrarily partition the page into units and then proceed to code each unit. To do this a uniform rectangular grid was drawn with each section in the grid having an area of one-fourth of a line of print. Such a subdivision of a page of text is shown in Figure 2.

Experience showed that the uniform grid method of partitioning a page of text had a number of drawbacks. Typically the format of a page of mathematics text did not conform well to the grid. Much of the time the natural divisions between types of messages on a page did not fit the arbitrary divisions imposed by the grid. The grid produced many subdivisions containing parts of two different types of messages and this made classification difficult. The grid also tended to divide single messages into a number of units and this made the task of coding the units unnecessarily repetitive.

Because of the problems inherent with the uniform grid, it was decided to use a method of subdivision which would more closely conform to the natural divisions between types of messages. This subdivision of a page of text is determined by first drawing boxes around each illustration, sentence, exercise, section heading, and blank space which appears on a

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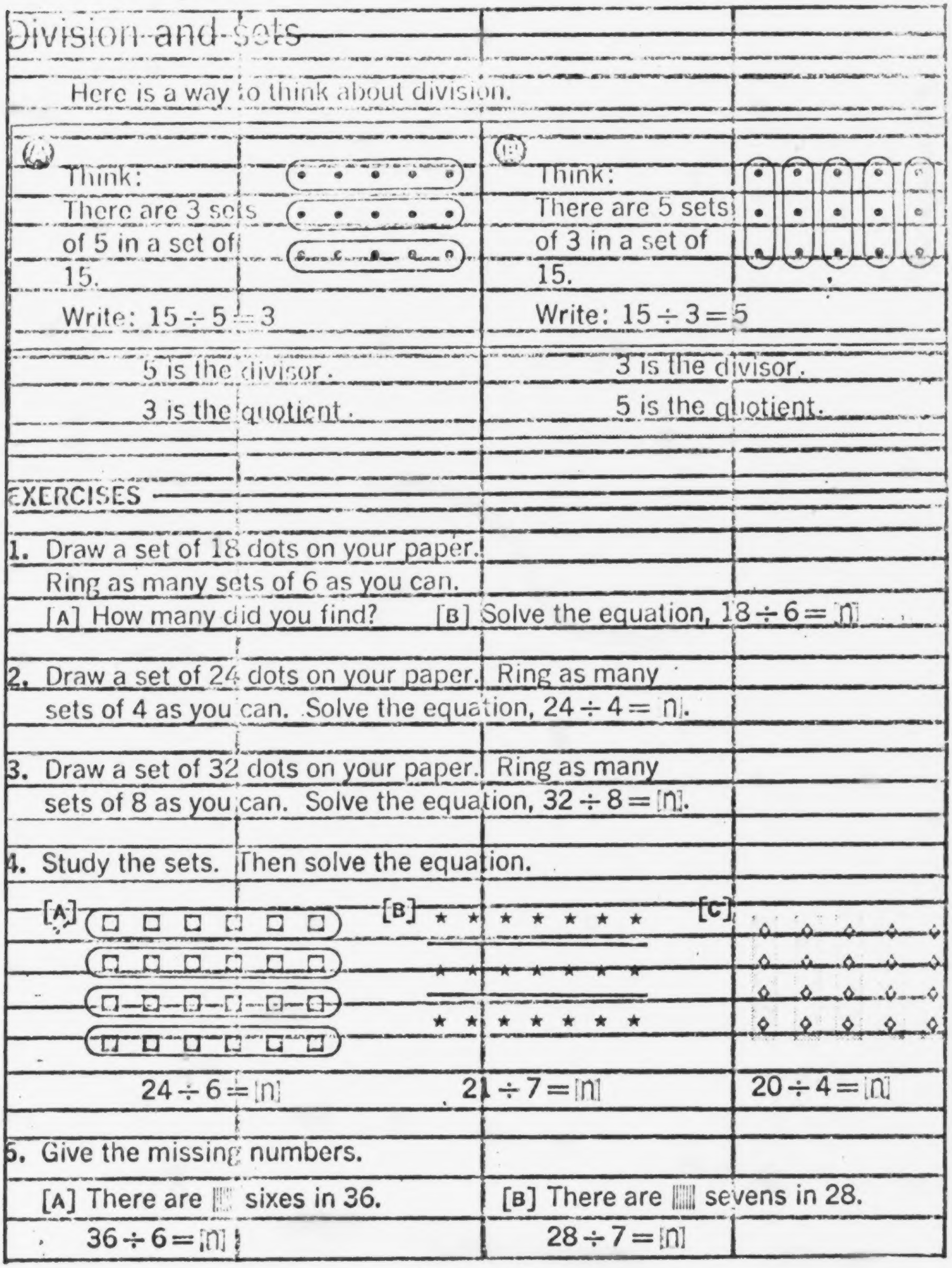

Division and sets

Here is a way to think about division.

Ⓐ Think:
There are 3 sets of 5 in a set of 15.
Write: 15 ÷ 5 = 3

Ⓑ Think:
There are 5 sets of 3 in a set of 15.
Write: 15 ÷ 3 = 5

5 is the divisor.
3 is the quotient.

3 is the divisor.
5 is the quotient.

EXERCISES

1. Draw a set of 18 dots on your paper. Ring as many sets of 6 as you can.
[A] How many did you find? [B] Solve the equation, 18 ÷ 6 = [n]

2. Draw a set of 24 dots on your paper. Ring as many sets of 4 as you can. Solve the equation, 24 ÷ 4 = [n].

3. Draw a set of 32 dots on your paper. Ring as many sets of 8 as you can. Solve the equation, 32 ÷ 8 = [n].

4. Study the sets. Then solve the equation.

[A] 24 ÷ 6 = [n]

[B] 21 ÷ 7 = [n]

[C] 20 ÷ 4 = [n]

5. Give the missing numbers.

[A] There are ▨ sixes in 36.
36 ÷ 6 = [n]

[B] There are ▨ sevens in 28.
28 ÷ 7 = [n]

Figure 2

Uniform Subdivision of Illustrative Passage page. Sentences are then further subdivided by separating the words from the mathematical symbols. Experience showed that this more natural method of subdivision produces a partition of a page which is easier to work with than the partition determined by the uniform grid. An example of a partitioned passage is given in Figure 3. This partition can be compared with the partitioning by the rectangular grid given in Figure 2.

Weighting Messages

Even though the uniform rectangular subdivision was rejected, a basic area unit was retained in the sense that when a CMMT analysis is made of a passage each subdivision is weighted in terms of its area. This unit of area is the area occupied by one-fourth of a line of print in the passage being analyzed. Its length is determined by one-fourth of the distance between the right and left margins of a full line of expository print. Its height is the distance between the lower edges of two successive lines of expository print.

Each subdivision in a coded passage is weighted to the nearest number of units which it occupies. Thus, if a subdivision covers the area of approximately two and one-fourth lines of print it receives a weight of nine. If it takes up the area of two-thirds of a line of print it receives a weight of three. Subdivisions taking up less than one-eighth of a line of print receive a weight of one.

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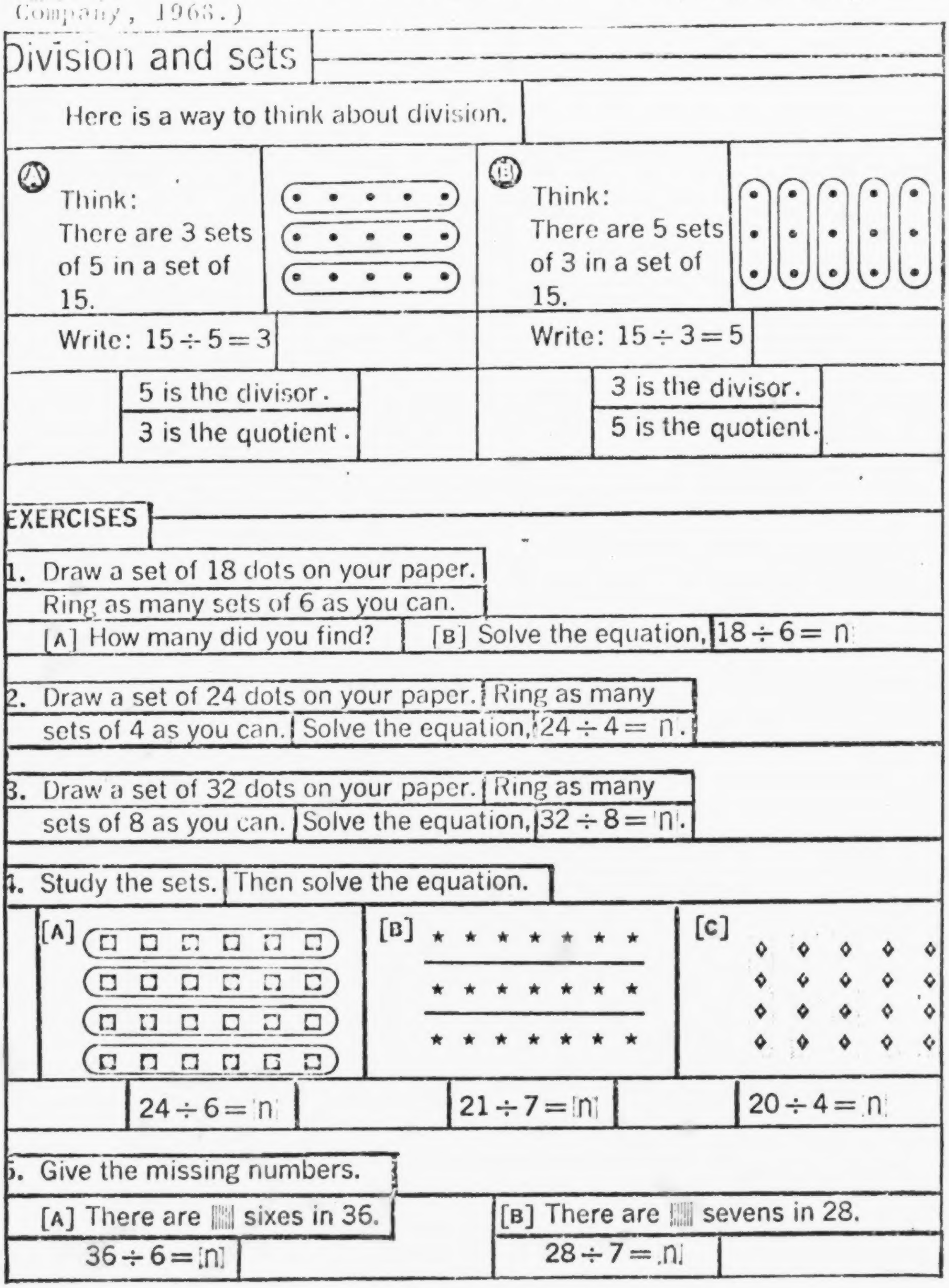


Division and sets

Here is a way to think about division.

Ⓐ Think: There are 3 sets of 5 in a set of 15.

Write: 15 ÷ 5 = 3

5 is the divisor.

3 is the quotient.

Ⓑ Think: There are 5 sets of 3 in a set of 15.

Write: 15 ÷ 3 = 5

3 is the divisor.

5 is the quotient.

EXERCISES

1. Draw a set of 18 dots on your paper. Ring as many sets of 6 as you can. [A] How many did you find? [B] Solve the equation, 18 ÷ 6 = n

2. Draw a set of 24 dots on your paper. Ring as many sets of 4 as you can. Solve the equation, 24 ÷ 4 = n.

3. Draw a set of 32 dots on your paper. Ring as many sets of 8 as you can. Solve the equation, 32 ÷ 8 = n.

4. Study the sets. Then solve the equation.

[A] 24 ÷ 6 = n

[B] 21 ÷ 7 = n

[C] 20 ÷ 4 = n

5. Give the missing numbers.

[A] There are ▨ sixes in 36.
36 ÷ 6 = n

[B] There are ▨ sevens in 28.
28 ÷ 7 = n

Figure 3

CMMT Partition of Illustrative Passage

In CMMT analysis, these weights are used to determine areas devoted to the various types of messages in a passage. The weighting of subdivisions in terms of area allows the method of subdividing a passage to be somewhat arbitrary with the weighting system correcting for any differences in subdivisions determined by different users.

Coding Passages

After a passage has been subdivided the messages in each subdivision are classified on both dimensions of the CMMT category system. This is done by recording a symbol of the form m-n in each subdivision, where m is the classification on dimension 1 and n is the classification on dimension 2 of the CMMT category system. Figure 4 is an example of a coded page of text.

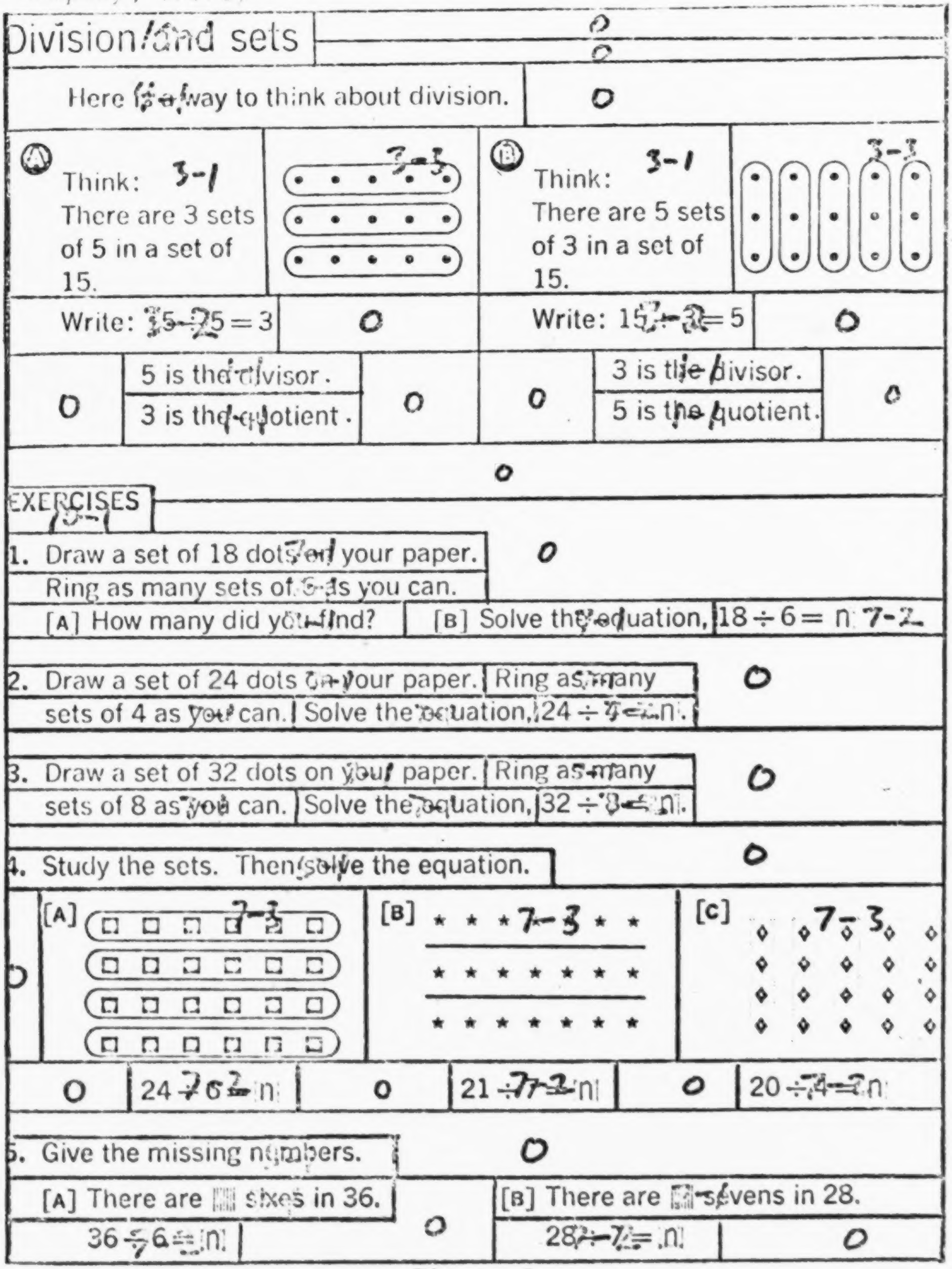
General Coding Procedures

Experience coding passages using the CMMT system led to the discovery of some general procedures and decision rules which proved useful. These are presented below in outline form as notes on using the CMMT technique.

Notes on Using Dimension 1 Categories.

A. Begin by mentally delineating between the blank spaces (0), exposition (1 - 4), messages requiring responses (5 - 9), and other material (10). This will make the task of making finer distinctions easier.

(Reprinted by permission. Elementary School Mathematics, Book 4, Eicholz and O'Daffer, Addison-Wesley Publishing Company, 1968.)



Division/and sets 0 0

Here is/a/way to think about division. 0

Ⓐ Think: 3-1 There are 3 sets of 5 in a set of 15. 3-3

Write: 15÷5 = 3 0

0 5 is the/divisor. 3 is the/quotient. 0

Ⓑ Think: 3-1 There are 5 sets of 3 in a set of 15. 3-3

Write: 15÷3 = 5 0

0 3 is the/divisor. 5 is the/quotient. 0

0

EXERCISES 3-1

1. Draw a set of 18 dots/on your paper. 0
Ring as many sets of 6 as you can.
[A] How many did you/find? [B] Solve the/equation, 18 ÷ 6 = n. 7-2

2. Draw a set of 24 dots on/your paper. Ring as/many sets of 4 as you/can. Solve the/equation, 24 ÷ 4 = n. 0

3. Draw a set of 32 dots on your/paper. Ring as/many sets of 8 as/you can. Solve the/equation, 32 ÷ 8 = n. 0

4. Study the sets. Then/solve the equation. 0

0 [A] 7-3 [B] 7-3 [C] 7-3

0 24 ÷ 6 = n 0 21 ÷ 7 = n 0 20 ÷ 4 = n

5. Give the missing numbers. 0

[A] There are ▒ sixes in 36. 0 [B] There are ▒ sevens in 28.

36 ÷ 6 = n. 28 ÷ 7 = n. 0

Figure 4

Coded Illustrative Passage

B. To rate exposition (1 - 4):

1. First identify definitions (1) and generalizations (2).
2. Of the remaining exposition, classify all messages giving specific instances of more general content as specific explanation (3).
3. Classify the remaining exposition as general explanation (4).

C. To rate messages requiring responses (5 - 9):

1. First identify procedural instructions (5).
2. Base decisions involving exercises and problems (6 - 9) on the processes involved. Do not base these decisions on the difficulty of the problem or exercise.

D. Illustrations as well as written messages must be classified on dimension 1. The classification of an illustrative message is determined by the same rules as the classification of written messages. If a diagram illustrates a definition it is classified as a definition, if it illustrates a proof it is classified as general explanation, etc.

E. When in doubt about how to classify a message reread the descriptions of the appropriate categories.

Notes on Using Dimension 2 Categories.

A. To decide if a written message in a unit is predominantly words or symbols, count the words necessary to say the message. If the symbols account for more than or the same number of these words as the ordinary English then classify the message as symbols (2). Otherwise, classify the message as words (1).

B. When classifying messages (3 - 7) identify the non-mathematical illustrations (6) first. Then make the remaining classifications. When in doubt reread the appropriate category descriptions.

Range of Application and the Sampling of Passages

The range of applicability for the CMMT technique was conjectured to be grade levels 4 through 12. This range was chosen since children in grades K through 3 seldom study mathematics through reading text materials and since most college mathematics textbooks are formal in nature and written for specialized audiences.

Typically mathematics textbooks are divided into large sections which cover a single broad topic or set of related topics. In turn, these sections are usually divided into short lessons. In each lesson a set of related concepts are presented. When sampling text passages for CMMT analysis entire lessons should be sampled. Only in this way can the organization of the material in a text be adequately described.

Description of CMMT Analysis

When a passage of text has been coded using the CMMT system, the information obtained may be displayed and analyzed in the following ways.

Listing

The sequence of messages in a passage may be displayed by listing them in the order in which they appear in the text. This order is determined by following the natural flow of the material through the subdivisions of the partition used in the classification process. The natural flow of the material is basically a left-to-right and top-to-bottom ordering with modifications to accommodate the format of the passage. In the list, the classifications for each dimension of the category system and the associated weight are recorded for each subdivision of the partition. The manner in which the list may be used to analyze the sequential nature of a passage is illustrated in Figure 5 which is the list for the illustrative passage coded in Figure 4.

Matrices

Matrices are used to analyze the interactions between categories in each dimension and the interaction between dimensions of the CMMT system. For interactions between categories within dimensions a matrix of the frequency with which a given message is followed by another given message is used. This yields a 10 × 10 matrix for dimension 1 and

		Dim 1	Dim 2		Weight
	Heading →	10	1		1
		0	0		3
Exposition	Intro. Statement →	4	1		2
		0	0		2
	Example followed by definition.	3	1	Word	4
		3	3	Illus.	4
		3	2	Symbol	1
		0	0	Sequence.	2
		1	1		2
		0	0		1
	Example followed by definition.	3	1	Word	4
		3	3	Illus.	4
		3	2	Symbol	1
		0	0	Sequence.	2
		1	1		2
		0	0		5
	Heading →	10	1		1
		0	0		7
Exercises	Exercise	7	1	Word	7
		7	2	Symbol.	1
		0	0		6
	Exercise	7	1	Word	5
		7	2	Symbol.	1
		0	0		6
	Directions →	5	1		2
		0	0		2
	Exercise	7	3	Illus.	5
		7	2	Symbol.	1
		0	0		1
	Exercise	7	3	Illus.	5
		7	2	Symbol.	1
	Directions →	5	1		2
	Exercise	7	1	Word	2
		7	2	Symbol.	1
		0	0		1
	Exercise	7	1	Word	2
		7	2	Symbol.	1
		0	0		1

(The weight indicates the number of area units of approximately one-fourth of a line of print devoted to each message recorded in the list.)

Figure 5

CMMT List for Illustrative Passage

a 7 × 7 matrix for dimension 2. The manner in which these within dimension matrices are used to analyze passages is illustrated below by the matrices for the illustrative passage coded in Figure 4.

Dimension 1
Categories

Dimension 1 Categories	1	2	3	4	5	6	7	8	9	10
1	2	0	1	0	0	0	0	0	0	1
2	0	0	0	0	0	0	0	0	0	0
3	2	0	16	0	0	0	0	0	0	0
4	0	0	1	1	0	0	0	0	0	0
5	0	0	0	0	2	0	2	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	2	0	35	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	1	0	0	1	0	0	0

Figure 6

Dimension 1 Interaction Matrix for Illustrative Passage

The notes which follow describe the dimension 1 interaction matrix given in Figure 6.

a. Entries in the matrix are in terms of units of one-fourth of a line of print. Entry n in row i and column j indicates units of type i were followed by units of type j, n times in the passage. Blank space is ignored.

b. The sum of the entries in the shaded area represents the number of shifts between messages requiring only reception and messages requiring responses. in the passage analyzed in the above matrix there was only one such shift, from category 10 to category 7.

c. Entries on the diagonal of the matrix represent numbers of successive repetitions of categories. The 16 in position (3,3) and the 35 in position (7,7) indicate that the passage analyzed in the above matrix consisted mainly of examples (category 3) and comprehension exercises (category 7).

d. Off diagonal entries in area A represent shifts between expository categories. There were four such shifts in the passage analyzed in the above matrix.

e. Off diagonal entries in area B represent shifts between categories requiring responses. In the passage analyzed there were four such shifts, all between directions (category 5) and comprehension type exercises (category 7).

f. Rows 2, 6, 8, and 9 contain all zeros, indicating that in the passage analyzed no generalizations were made, there were no developmental exercises, and no exercises or problems requiring application or higher mathematical processes.

Dimension 2 Categories		1	2	3	4	5	6	7
	1	29	4	3	0	0	0	0
	2	6	0	2	0	0	0	0
	3	0	5	18	0	0	0	0
	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

A

B

Figure 7

Dimension 2 Interaction Matrix for Illustrative Passage

The notes which follow describe the dimension 2 interaction matrix given in Figure 7.

a. Entries in this matrix are in terms of weights. Entry n in row i and column j indicates units of type i were followed by units of type j, n times in the passage. Blank space is ignored.

b. The sum of the entries in the shaded area of the matrix represents the number of shifts between written materials and illustration. In the passage analyzed in the above matrix there were 10 such shifts.

c. Entries on the diagonals represent numbers of successive repetitions of categories. The 29 in position (1,1) and the 18 in position (3,3) indicate that the majority of the material consisted of word messages (category 1) and illustrative representations of abstract ideas (category 3).

d. Off-diagonal entries in area A represent shifts between symbols and words within the written message. There were 10 such shifts in the passage analyzed above.

e. Off-diagonal entries in area B represent shifts between types of illustrations. There were no such shifts in the passage analyzed above.

To analyze interactions between dimensions of the CMMT category system, a 10 × 7 matrix is used. The between dimension interaction matrix for the passage coded on page 21 is given below.

Dimension 1
Categories

		1	2	3	4	5	6	7
	1	4	0	0	0	0	0	0
	2	0	0	0	0	0	0	0
	3	8	2	8 A	0	0	0	0
	4	2	0	0	0	0	0	0
Dimension 2 Categories	5	4	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	16	7	15 B	0	0	0	0
	8	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	2	0	0	0	0	0	0

Figure 8

Between Dimension Interaction Matrix for Illustrative Passage

The following notes describe the between dimension interaction matrix given in Figure 8.

a. Entries in this matrix are made in terms of weights. Entry n in row i and column j indicates units of type i on dimension 1 were of type j on dimension 2, n times in the passage.

b. Entries in area A show the presentation mode used for the exposition. The matrix above shows that the exposition in the passage analyzed consisted of four units of definition presented in words (row 1), 18 units of specific explanation presented in words, symbols, and illustrations (row 3) and two units of general explanation in words (row 4).

c. Entries in area B show the presentation mode used for exercises and problems. The matrix shows that in the passage analyzed four units were word directions (row 5) and 38 units were comprehension exercises presented in words, symbols, and illustrations (row 7).

Proportions

The list and the matrices are used to describe sequential aspects of mathematics passages. Proportions computed in terms of the weights of the various messages are used to describe quantitative aspects of the organization of passages. The defining formulas for the proportions are given below. In the defining formulas, $c_{n,i}$ = the number of units devoted to category i on dimension n.

Proportion of blank space in total passage.

$$c_{1,0} \Big/ \sum_{0}^{10} c_{1,i}$$

Proportion of category j in non-blank material dimension 1.

$$c_{1,j} \Big/ \sum_{1}^{10} c_{1,i}$$

Proportion of category j in non-blank material dimension 2.

$$c_{2,j} \Big/ \sum_{1}^{7} c_{2,i}$$

Proportion of exposition in non-blank material.

$$\sum_{1}^{4} c_{1,i} \Big/ \sum_{1}^{10} c_{1,i}$$

Proportion of content development in non-blank material.

$$\left(\sum_{1}^{4} c_{1,i} + c_{1,6}\right) \Big/ \sum_{1}^{10} c_{1,i}$$

Proportion of exercises in content development.

$$c_{1,6} \Big/ \left(\sum_{1}^{4} c_{1,i} + c_{1,6}\right)$$

Proportion of exposition in content development.

$$\sum_{1}^{4} c_{1,i} \Big/ \left(\sum_{1}^{4} c_{1,i} + c_{1,6}\right)$$

Proportion requiring responses in non-blank material.

$$\sum_{5}^{9} c_{1,i} \Big/ \sum_{1}^{10} c_{1,i}$$

Proportion of mathematics illustration in non-blank material.

$$\sum_{3}^{5} c_{2,i} \Big/ \sum_{1}^{7} c_{2,i}$$

Proportion of illustration in non-blank material.

$$\sum_{3}^{7} c_{2,i} \Big/ \sum_{1}^{7} c_{2,i}$$

The proportions computed for the illustrative passage coded on page 21 are given in Table 1.

The proportions in Table 1 indicate that in the passage analyzed the non-blank material consisted mainly of specific explanation or examples (category 3 of dimension 1) and comprehension exercises (category 7 of dimension 1). Slightly over one-half of the material consisted of word messages (category 1 of dimension 2), about one-third consisted of abstract illustrations (category 3 of dimension 2), and the remainder consisted of mathematical symbols (category 2 of dimension 2). The passage can be characterized as consisting of about one-third exposition and about two-thirds exercises and problems. All of the content development material was presented in an expository manner.

A computer program was written to do the work of deriving the CMMT analysis for a passage from a rater's codings. The

Table 1

CMMT Proportions for Illustrative Passage.

Proportion of blank space in total passage	.409

Proportion of each category in non-blank material.

Category	Dimension 1	Dimension 2
1	.060	.537
2	.000	.119
3	.269	.343
4	.030	.000
5	.030	.000
6	.000	.000
7	.552	.000
8	.000	----
9	.000	----
10	.030	----

Proportion of exposition in non-blank material	.358
Proportion of content development in non-blank material	.358
Proportion of exercises in content development	.000
Proportion of exposition in content development	1.000
Proportion requiring responses in non-blank material	.612
Proportion of mathematics illustration in non-blank material	.343
Proportion of illustration in non-blank material	.343

output of this program includes the list, the matrices, and the proportions for the analyzed passages. The Fortran statements for the program and data set up for using it are included in Appendix E.

A Method of Estimating Rater Reliability

A method of estimating the reliability of raters using the CMMT technique has been adapted from a method of estimation used in interaction analysis (Flanders, 1960, p 161). This method first proposed by Scott (1955), is based on an adjusted proportion of agreement between raters. The estimate can be made whenever two or more raters code the same passage. It is relatively easy to compute and interpret. Thus, the estimate is more practical than other possible reliability coefficients based on analysis of variance or on correlational procedures.

Scott's coefficient is called "pi" and is determined by the formula

$$\pi = \frac{P_o - P_e}{1 - P_e} .$$

P_o is the proportion of agreement between two raters. P_e is the proportion of agreement between two raters expected by chance which is found by squaring the proportion of classifications made by the raters in each category and summing these over all categories. That is,

$$P_e = \sum_1^k P_i^2 ,$$

where P_i is the proportion classified as category i out of k categories. The coefficient π can be expressed in words as the amount that two observers exceeded chance agreement divided by the amount that perfect agreement exceeds chance.

When determining a between-rater reliability estimate for more than two raters, the Scott coefficient is determined for each pair of raters and the average of all such pairs is taken as the reliability estimate. When investigating the reliability of raters using the CMMT technique, an estimate is made for each dimension of the category system. The computational formulas are given in Figure 9.

Reliability Estimate for Two Raters on Dimension n.	$\pi_n = \frac{P_o - P_e}{1 - P_e}$
Proportion of Agreement between Two Raters on Dimension n.	$P_o = \frac{\text{(Number of agreements on dimension n)}}{\text{(Total number of messages)}}$
Proportion of Agreement Expected by Chance between Two Raters on Dimension n.	$P_e = P_i^2$ where $P_i = \frac{\text{(Number of dimension messages classed as i by first rater plus the number classed as i by second)}}{\text{2 (total number of messages)}}$

a. Blank space (category 0) is ignored in reliability computations.

b. For more than two raters compute all pairs of coefficients and average.

Figure 9

Method of Computing between-rater Reliability Estimates.

The Scott coefficient for estimating reliability may be used to determine a within-rater reliability coefficient by computing the proportion of the time a rater agrees with himself on two trials separated by a period of time. A Scott coefficient of .85 or above is considered to be a reasonable level of performance for raters using interaction analysis (Flanders, 1960, p 166). Acceptable levels of performance using the CMMT technique will have to be determined by research. The computer program found in Appendix F is used to compute Scott coefficients from the ratings of observers.

CHAPTER IV

PROCEDURES FOR THE EMPIRICAL STUDIES

The main purposes of the empirical studies were

1. To investigate the validity of the CMMT technique as a means of studying the organization of mathematics text.
2. To investigate the reliability of raters trained in the use of the CMMT technique.

This chapter is devoted to describing the procedures which were used to achieve each of these purposes.

The Validity Study

Validity was studied in both a descriptive and a statistical manner. A number of passages were chosen from mathematics textbooks and submitted to CMMT analysis by the investigator. These passages were used to demonstrate how CMMT analysis may be used to describe the organization of mathematics text. The data from the CMMT analyses were also submitted to correlational analysis and to analysis of variance in order to show further evidence of validity of the CMMT technique.

The specific validity objectives were

1. To demonstrate that CMMT analysis can be used to

describe the organization of existing mathematics text passages.

2. To demonstrate that CMMT analysis reflects similarities and differences between passages with known organizational similarities and differences.
3. To determine if CMMT analyses of entire sections of textbooks can be closely approximated by sampling a small number of CMMT sample passages.
4. To determine if the CMMT analyses of passages sampled within the same textbook are more closely related than the CMMT analyses of passages sampled from different textbooks.
5. To determine if the data obtained from CMMT analysis may be used to identify significant differences in organization between different mathematics textbooks.

The Materials Studied

The CMMT technique was designed to be applied to mathematics materials at approximately grade levels 4 through 12. Hence, questions of validity needed to be studied at a range of grade levels using various mathematical topics. The materials used in the study were grouped into three sets. The levels and corresponding topics for each set were

Set 1. Upper Elementary Materials (grades 4 - 6)
Arithmetic, Geometry, Introductory Algebra.

Set 2. Junior High School Materials (grades 7 - 9)
Arithmetic, Algebra, Geometry, Probability.

Set 3. Senior High School Materials (grades 10 - 12)
Algebra, Geometry, Trigonometry, Algebraic
Structure.

Twelve textbooks were chosen on a rational basis to obtain samples of all the topics and levels indicated above. Eight of the twelve textbooks were published by eight different commercial publishers. The other four books were written by four different mathematics curriculum groups. This selection procedure was used to insure coverage of a variety of topics, levels, and text types. No attempt was made to generalize about a particular publisher's or study group's textbooks in this study.

The textbooks studied were:

Set 1: B1. Elementary School Mathematics, Book Four, by Eicholz and O'Daffer, C 1968 by Addison-Wesley Publishing Company. (Grade level 4)

B2. Learning Mathematics, by Deans, et al., C 1968 by the American Book Company. (Grade level 5)

B3. Exploring Elementary Mathematics, Five, by Keedy, et al., C 1970 by Holt, Rinehart and Winston, Inc. (Grade level 5)

B4. Six, Modern School Mathematics, by Duncan, et al., C 1967 by Houghton Mifflin Company. (Grade level 6)

Set 2: B5. Seeing Through Mathematics, Book One, by Van Engen, et al., C 1962 by Scott, Foresman and Company.

(Grade level 7)

B6. Mathematics for Junior High, Volume 2, Part II, by the School Mathematics Study Group, C 1961 by Yale University Press.

(Grade level 8)

B7. Unified Modern Mathematics, Course II, Part I, from the Secondary School Mathematics Curriculum Improvement Study, C 1969 by Teachers College, Columbia University.

(Grade level approximately 8)

B8. Contemporary Algebra, Book One, by Smith, et al., C 1962 by Harcourt, Brace and World, Inc.

(Grade level 9)

Set 3: B9. Book Six, Relations, from the Comprehensive School Mathematics Program, C 1971 by CEMREL, Inc.

(Grade level approximately 10)

B10. High School Mathematics, Unit 6, Geometry, by the University of Illinois Committee on School Mathematics, C 1960 by University of Illinois Press.

B11. Contemporary Algebra, Second Course, by Mayor and Wilcox, C 1965 by Prentice-Hall, Inc.
(Grade level 11)

B12. Algebra with Trigonometry, by Fehr, et al., C 1963 by D. C. Heath and Company.
(Grade level 12)

From each of these twelve textbooks one experimental section was selected on a rational basis to cover the topics listed above for the various sets of material. The experimental sections, including 99 passages covering 310 pages, served as data sources for the study of the validity objectives. The sections and corresponding topics are listed below.

ES1. Arithmetic
(Chapter 3 of B1: Multiplication and Division, 14 passages, pages 72-87)

ES2. Geometry
(Chapter 7 of B2: Geometry, 9 passages, pages 229-242)

ES3. Arithmetic
(Chapter 2 of B3: Addition and Subtraction, 13 passages, pages 31-51)

ES4. Introductory Algebra
(Chapter 5 of B4: Statements, 11 passages, pages 130-141)

ES5. Arithmetic
(Unit 7 of B5: The Rational Numbers of Arithmetic, 9 passages, pages 308-327)

ES6. Probability
(Chapter 8 of B6: Probability, 6 passages, pages 311-345)

ES7. Geometry
(Chapter 3 of B7: An Introduction to Axiomatic Affine Geometry, 6 passages, pages 120-173)

ES8. Algebra
(Chapter 4 of B8: Operations with Algebraic Expressions, 11 passages, pages 119-135)

ES9. Algebraic Structure
(Chapter 2 of B9: Ordered Pairs and Relations, 4 passages, pages 27-69)

ES10. Geometry
(Section 6.07 of B10: Similarity, 5 passages, pages 6-179 to 6-218)

ES11. Algebra
(Chapter 7 of B11: Exponents and Radicals, 10 passages, pages 201-216)

ES12. Trigonometry
(Chapter 12 of B12: Trigonometric Functions, 7 passages, pages 427-448)

A focus passage representing a specific instance of the topics and levels to be studied was chosen on a rational basis from each of the twelve experimental sections. Some of these passages served as data sources for studying how CMMT analysis could be used to describe the organization of existing text. Others provided the mathematical material from which pairs of passages were contrived for studying how CMMT analysis can be used to compare similar passages. The focus passages were also used as practice and criterion passages in the reliability study discussed in the next section. The various focus passages and corresponding topics are listed below.

FP1. Division and Sets
(Page 73 of B1)

FP2. Circumference
(Pages 231-233 of B2)

FP3. Commutative Property of Addition
(Page 32 of B3)

FP4. Using Equations and Inequalities
(Pages 136-137 of B4)

FP5. Multiplication of Rational Numbers
(Pages 324-325 of B5)

FP6. Probability of A or B
(Pages 328-332 of B6)

FP7. Some Logical Consequences of the Affine Axioms
(Pages 128-135 of B7)

FP8. Expressions with Grouping Symbols
(Page 132 of B8)

FP9. Cartesian Products and Relations
(Pages 47-53 of B9)

FP10. Mean Proportional and the Pythagorean Theorem
(Pages 6-203 to 6-207 of B10)

FP11. Fractional Exponents
(Pages 210-211 of B11)

FP12. Extending the Abs and Ord Functions
(Pages 431-432 of B12)

From each of six focus passages a pair of contrived passages was written. Each contrived passage covered the same mathematical material as the focus passage on which it was based. The passages in a contrived pair were written following different organizational plans and were approximately the same length. The twelve contrived passages were used to study how CMMT analysis reflects similarities and differences between passages with known organizational similarities and differences. The organizational plans followed in constructing the contrived passages are listed next.

1. Plans of the contrived passages based on FP2.

- CP2a:
  - Dimension 1:
    - High proportion of developmental exercises.
    - Low proportion of explanation.
    - Integration of exercises with expository.
    - Other aspects constant with CP2b.
  - Dimension 2: Constant with CP2b.
- CP2b:
  - Dimension 1:
    - No developmental exercises
    - High proportion of explanation.
    - Exposition separated from exercises.
    - Other aspects constant with CP2a.
  - Dimension 2: Constant with CP2a.

2. Plans of the contrived passages derived from FP4.

- CP4a:
  - Dimension 1:
    - High proportion of examples.
    - Low proportion of general explanation.
  - Dimension 2: Use of motivational illustration.
  - Other aspects constant with CP4b.
- CP4b:
  - Dimension 1:
    - No examples.
    - High proportion of general explanation.
  - Dimension 2:
    - Use of mathematical illustrations.
  - Other aspects constant with CP4a.

3. Plans of contrived passages derived from FP6.

- CP6a:
  - Dimension 1: Constant with CP6b.
  - Dimension 2:
    - High proportion of symbols and illustrations.
    - Low proportion of words.
- CP6b:
  - Dimension 1: Constant with CP6a.
  - Dimension 2:
    - Low proportion of symbols and illustrations.
    - High proportion of words.

4. Plans of contrived passages based on FP7.

- CP7a:
  - Dimension 1:
    - General explanation followed by generalization sequences.
    - Other aspects constant.
  - Dimension 2: Constant with CP7a.
- CP7b:
  - Dimension 1:
    - Generalization followed by general explanation sequences.
    - Other aspects constant.
  - Dimension 2: Constant with CP7a.

5. Plans of contrived passages based on FP9.

- CP9a:
  - Dimension 1:
    - Low proportion of definition, generalization, and general explanation.
    - High proportion of examples.
    - Exercises integrated with exposition.
  - Dimension 2: Constant with CP9a.
- CP9b:
  - Dimension 1:
    - High proportion of definition, generalization, and general explanation.
    - Low proportion of examples.
    - Exposition separated from exercises.
  - Dimension 2: Constant with CP9b.

6. Plans of contrived passages based on FP10.

- CP10a:
  - Dimension 1:
    - High proportion of exposition.
    - Low proportion of exercises and problems.
  - Dimension 2:
    - Illustrations grouped separate from written text.
  - Other aspects constant with CP10b.
- CP10b:
  - Dimension 1:
    - Low proportion of exposition.
    - High proportion of exercises and problems.
  - Dimension 2:
    - Illustrations integrated with written text.
  - Other aspects constant with CP10a.

The choice of the organizational plans followed in the preparation of a given contrived pair of passages was made on a rational basis depending on the related focus passage. For example, if the focus passage contained mostly manipulative algebraic material no attempt was made to contrive passages with differing illustrative material. Thus, the material in the focus passages limited to some extent the organizational variables which could be manipulated in the contrived passages. No attempt was made in this study to exhaust or even adequately sample all possible organizational variables which could be investigated. The contrived passages served only to demonstrate the feasibility of using CMMT analysis to describe the organization of passages.

The chart in Figure 10 summarizes the selection and preparation of the materials used in the validity portion of the study. The labeling scheme used here for materials will be used throughout the remaining chapters to refer to materials.

Method of Studying the Materials

All of the materials described in the previous section were submitted to CMMT analysis by the investigator. These analyses were used to attain the validity objectives given on page 2.

The focus passages FP1, FP3, FP5, FP8, FP11, and FP12 were used to demonstrate that CMMT analysis can be used to describe the organization of mathematics passages. These

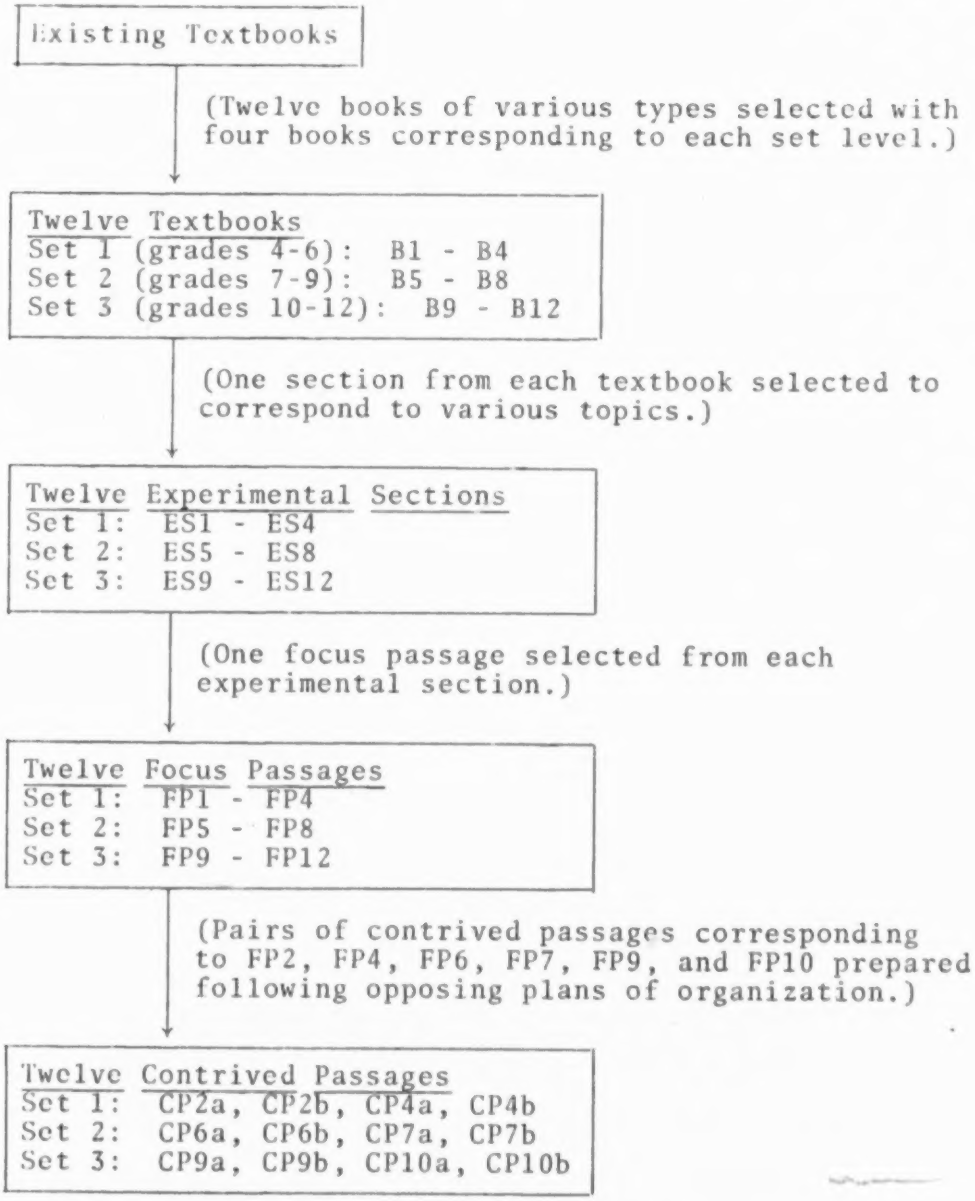

Existing Textbooks
(Twelve books of various types selected with four books corresponding to each set level.)
Twelve Textbooks
Set 1 (grades 4-6): B1 - B4
Set 2 (grades 7-9): B5 - B8
Set 3 (grades 10-12): B9 - B12
(One section from each textbook selected to correspond to various topics.)
Twelve Experimental Sections
Set 1: ES1 - ES4
Set 2: ES5 - ES8
Set 3: ES9 - ES12
(One focus passage selected from each experimental section.)
Twelve Focus Passages
Set 1: FP1 - FP4
Set 2: FP5 - FP8
Set 3: FP9 - FP12
(Pairs of contrived passages corresponding to FP2, FP4, FP6, FP7, FP9, and FP10 prepared following opposing plans of organization.)
Twelve Contrived Passages
Set 1: CP2a, CP2b, CP4a, CP4b
Set 2: CP6a, CP6b, CP7a, CP7b
Set 3: CP9a, CP9b, CP10a, CP10b

Figure 10

Materials Used in the Validity Study.

passages are included in Appendix C. The CMMT analyses of these passages were studied in detail. The manner in which the analyses reflected the organization of the passages is discussed in Chapter 5.

The CMMT analyses of the contrived passages were compared and contrasted to see how they reflected the organizational plans of the passages. The manner in which the CMMT analyses described the planned similarities and differences between the contrived passages in each pair is discussed in detail in the next chapter. The contrived passages are included in Appendix D.

To determine if CMMT analyses of entire sections of textbooks can be closely approximated by sampling a small number of CMMT sample passages, successive correlations were made between each entire experimental section and increasingly larger random samples of passages from the same section. For these correlations, 25 items of data readily available from the CMMT analyses of the entire sections and the samples were used. These items of data are listed below.

1. The average number of non-zero entries in the dimension 1 interaction matrices.
2. The percentage of off-diagonal entries in the dimension 1 interaction matrix.
3. The percentage of shifts between exposition and exercises in the dimension 1 interaction matrix.

4. The average number of non-zero entries in the dimension 2 interaction matrices.
5. The percentage of off-diagonal entries in the dimension 2 interaction matrix.
6. The percentage of shifts between written and illustration in the dimension 2 interaction matrix.
7. The average number of non-zero entries in the between-dimension interaction matrices.
8. The percentage of blank space.

9.
.
. } The percentage in each category (1 - 10) on dimension 1.
.
18.

19.
.
. } The percentage in each category (1 - 7) on dimension 2.
.
25.

The following procedure was used in making these successive correlations. A passage was selected at random from a given section and the above data from the CMMT analysis of the entire section was correlated with the corresponding data for the passage. Then another passage was randomly selected from the remaining passages in the section, combined with the first passage for CMMT analysis and the same correlational procedure was repeated. The process was continued until all passages in the section were sampled. It was determined how many passages it was necessary to sample in order to reach and maintain correlations of .90 and .95

between the sample and the entire section. The entire procedure was repeated for each of the twelve experimental sections.

To determine how closely the CMMT analyses of passages sampled within the same textbook are related, each passage in an experimental section was correlated with each other passage in the section using the 25 items of CMMT data listed above. The average of these within-textbook correlations was computed for each experimental section. A λ^2 test suggested by Ostle (1963, p. 227) was used to test for significant differences among the twelve average within-textbook correlations.

In addition, the average correlation between all possible pairs of twelve passages, one chosen at random from each experimental section, was computed using the 25 items of CMMT data. This average between-textbook correlation was compared with each of the twelve within-textbook average correlations to determine if the within-textbook passages correlated significantly higher than the between-textbook passages. The λ^2 test mentioned above was also used for making these comparisons.

To determine if the data obtained from CMMT analysis could be used to identify significant differences in organization between different textbooks, an analysis of variance model was used. The variables investigated in this manner were variables which could be readily obtained from the CMMT analyses of the passages. These variables were chosen to represent the types of CMMT variables it is possible to study. They do not exhaust all possible CMMT variables which could be investigated. The sixteen variables which were investigated are listed next.

1. Total units per passage.
2. Number of non-zero entries in the dimension 1 interaction matrix per passage.
3. Percentage of off-diagonal entries in the dimension 1 interaction matrix per passage.
4. Percentage of shifts between exposition and exercises per passage.
5. Number of non-zero entries in the dimension 2 interaction matrix per passage.
6. Percentage of off-diagonal entries in the dimension 2 interaction matrix per passage.
7. Percentage of shifts between written and illustration per passage.
8. Number of non-zero entries in the between dimension interaction matrix per passage.
9. Percentage of blank space per passage.
10. Percentage of exposition per passage.
11. Percentage of content development per passage.
12. Percentage of activities in content development per passage.
13. Percentage of exposition in content development per passage.

14. Percentage requiring responses per passage.

15. Percentage of mathematical illustration per passage.

16. Percentage of illustration per passage.

Each of the above variables was submitted to a one-way unequal cell analysis of variance as illustrated in Figure 11 (Winer, 1962, pp. 96-104). In this design, passages within textbooks are used in place of the more typical subjects within treatments. Thus, in the analysis of variance $MS_{\text{between treatment}}$ is interpreted as $MS_{\text{between textbooks}}$ and $MS_{\text{within treatments}}$ is interpreted as $MS_{\text{within textbooks}}$. When significant differences were found between the experimental textbooks on a given variable, the Numan-Keuls Sequential Range Test (Winer, 1962, p. 80) was used to obtain an ordering of the textbooks on the variable.

	B1	B2	. . .	B12
	$x_{1,1}$	$x_{1,2}$		$x_{1,12}$
	.	.		.
	.	.		.
	.	.		.
Variable m	$x_{i,1}$	$x_{i,2}$		$x_{i,12}$
	.	.		.
	.	.		.
	.	.		.
	$x_{13,1}$	$x_{9,2}$		$x_{7,12}$

$x_{i,j}$ = variable m CMMT score on passage i of textbook Bj.

Figure 11

Analysis of Variance Design for Identifying Differences between Textbooks.

The Reliability Study

To study the reliability of raters trained in the use of the CMMT technique, a rater training booklet and a booklet of criterion passages were prepared. The materials for these booklets were drawn from the validity materials described in a previous section. Three groups of subjects studied the training booklet individually and on two occasions coded the passages in the criterion booklet. These passages were scored in terms of the proportion of agreement of each subject with the most frequent ratings of the meassages. A repeated measures analysis of variance model was used to analyze the differences between groups and between the rating difficulties of the criterion passages. The five subjects in each group who received the highest scores were considered proficient in applying the CMMT technique and were defined as experts in their groups. Within- and between-rater reliability coefficients were determined for the experts in each group.

The specific reliability objectives were:

1. To determine if there are significant differences between groups of raters with differing academic background and experience.
2. To determine if there are significant differences in the level of rating difficulty between different types of passages.

3. To determine between-rater reliability coefficients for raters who obtained proficiency in the CMMT technique.
4. To determine within-rater reliability coefficients for raters who obtained proficiency in the CMMT technique.
5. To determine a reliability estimate for ratings made by the investigator in the validity portion of the study.

Materials

A training booklet was written for subjects to use in training themselves in the use of the CMMT technique. The booklet consisted of an introduction to the CMMT categories, thorough descriptions of the categories, procedural suggestions for applying the CMMT system, and many-keyed practice passages. Descriptive material in the training booklet was based on the developmental work discussed in Chapter 3. The practice passages included were selected from the textbooks studied in the validity portion of the study and represented a range of subject matter and grade levels. Each practice passage was followed by a coded version of the passage so that subjects could obtain some feedback as they learned the system. A copy of the training booklet is included in Appendix A.

The booklet of criterion passages, included in Appendix B, consisted of six partitioned passages to be coded by the subjects. Three of the six criterion passages were focus passages FP1, FP8, and FP12 of the validity study. The other three criterion passages were contrived passages CP2b, CP6a, and CP10b. One focus passage and one criterion passage corresponded to each of the three levels of materials studied in the validity study. The criterion passages ranged in subject matter over arithmetic, algebra, geometry, probability, and trigonometry. None of the criterion passages was taken from the same experimental section as practice passages used in the training booklet. The order in which the criterion passages appeared was randomized for each booklet. This was done to avoid any possible order effect in the analysis of passage rating difficulty.

Subjects

Three groups of subjects from different populations were used as CMMT raters in the reliability study. The groups were

Group 1. Doctoral students and faculty members in mathematics education.

Group 2. Experienced secondary mathematics teachers with or near a Master's degree.

Group 3. University seniors preparing to become teachers of secondary mathematics.

Subjects in group 1 consisted of people working in mathematics education or recent graduates in mathematics education from Purdue University, the University of Georgia, and California State Polytechnic College. Subjects in group 2 consisted of members of Purdue University's 1971 N.S.F. Summer Institute for secondary mathematics teachers. Subjects in group 3 consisted of Purdue students in Education 304S which is a course in the methods of teaching secondary school mathematics offered during the professional semester of the senior year. All subjects participated on an invited basis. There were 15 subjects in each group.

Design of the Reliability Study

Each group of subjects used the training booklet to learn the use of the CMMT technique. Subjects were directed to train themselves by studying the booklet. After this self-training, subjects were directed to code the criterion passages on two different occasions separated by an interval of at least four weeks. Subjects worked individually and were instructed to spend a maximum of four hours on the training booklet and one hour on each rating of the criterion passages. Subjects were allowed the use of the training booklet for reference during the rating of the criterion passages. The experimental procedure is illustrated in Figure 12.

Method of Analyzing the Reliability Data

To obtain between-rater scores, each subject's criterion passages for trial 1 were scored in terms of the proportion of agreement with criterion scores based on the most frequent classifications of the messages in these passages. To

Subjects	Subjects individually trained using the training booklet.	Trial 1: Subjects rated criterion passages.	Four week waiting period.	Trial 2: Subjects rated criterion passages again.
1→ Group 1 . . . 15→			▨	
1→ Group 2 . . . 15→			▨	
1→ Group 3 . . . 15→			▨	

Figure 12

Experimental Procedure for the Reliability Study.

obtain within-rater scores each subject's criterion passages were also scored in terms of the proportion of messages rated the same on both trials. In addition to an overall score of each type, a between- and within-rater score on each CMMT dimension was obtained for each subject on each passage. A computer program was written to score the criterion passages and to compute the KR20 reliability coefficient for each CMMT dimension of each passage. This program is included in Appendix G.

To assess differences between groups and between the rating difficulty of passages on each of the CMMT dimensions in terms of each scoring procedure, four repeated measures analyses of variance were performed. The design for these analyses of variance which is given in Winer (1962, pp. 302-318) is illustrated in Figure 13. Appropriate a posteriori tests were made when differences among means were found.

	Passages					
	Level 1		Level 2		Level 3	
Subjects	FP1	CP2b	FP8	CP6a	FP12	CP10b
Group 1 1→ . . . 15→						
Group 2 1→ . . . 15→						
Group 3 1→ . . . 15→						

Level 1 (grades 4 - 6).
Level 2 (grades 7 - 9).
Level 3 (grades 10 - 12).
Analysis 1: Dimension 1 between-rater scores.
Analysis 2: Dimension 2 between-rater scores.
Analysis 3: Dimension 1 within-rater scores.
Analysis 4: Dimension 2 within-rater scores.

Figure 13

Analysis of Variance Design for Assessing Passage Rating Difficulty and Differences between Groups.

For a number of reasons, high levels of performance were not expected of all subjects. Subjects could not be expected to be highly motivated in this situation. They were not required to spend a great deal of time training considering the complexity of the task. Further they had no opportunity to ask questions or to receive any other type of live feedback. Hence, it was not expected that all subjects would become proficient in the technique with the training they received. In order to estimate reliability, subjects obtaining proficiency were needed. In an attempt to obtain such subjects, the top five subjects in each group were selected on the basis of average overall first trial scores. These experts from the respective groups were used to determine reliability estimates.

Between- and within-rater reliability coefficients for each group of experts were determined for each criterion passage on each dimension of the CMMT system. This was done by computing the average Scott reliability coefficients described in Chapter 3 for each group of experts and each passage. A summary of all reliability coefficients computed for each group of experts on each passage is given in Figure 14.

To complete the reliability study a check was made of the reliability of the investigator. This was done by computing the proportion of agreement between the investigator and the between-rater criterion ratings on each criterion

		Between-rater	Within-rater
Scott Coefficients	Dimension 1	$\frac{\sum_{i<j} \pi_{1ij}}{10}$	$\frac{\sum_{k=1}^{5} \pi_{1k}}{10}$
	Dimension 2	$\frac{\sum_{i<j} \pi_{2ij}}{10}$	$\frac{\sum_{k=1}^{5} \pi_{2k}}{10}$

π_{mij} = between-rater Scott coefficient for dimension m and expert pair (i,j).

π_{mk} = within-rater Scott coefficient for dimension m and expert k.

Figure 14

Summary of Reliability Coefficients Computer for Each Group of Experts on Each Criterion Passage.

passage. A Scott reliability coefficient was also computed between the investigator and the criterion ratings on each criterion passage. These coefficients were used to infer the reliability of the ratings made by the investigator in the validity study.

CHAPTER V

RESULTS OF THE EMPIRICAL STUDIES

This research concerned the development of the CMMT technique for studying presentation variables in mathematics text. The purpose of the empirical studies was to investigate the validity and reliability of the CMMT technique. The results of these studies are presented in this chapter.

Results of the Validity Study

The validity results are both descriptive and statistical in nature. The descriptive results illustrate how CMMT analysis may be used to describe and compare mathematics text passages. The statistical results concern the number of passages which provide an adequate sample from a textbook section for CMMT analysis, correlational comparisons of the CMMT analyses of passages within and between textbooks, and the identification by CMMT analysis of presentation variables on which textbooks can be differentiated.

Descriptive Results for the Focus Passages

In order to illustrate how CMMT analysis may be used to describe mathematics text passages, focus passages FP1, FP3, FP5, FP8, FP11, and FP12 are presented in detail. These passages are included in Appendix C. In this section, a short description of the level and content of each of the passages is given. Each description is followed by the CMMT analysis of the passage described, which in turn is followed by a brief discussion of what the analysis shows.

Focus Passage FP1. This passage dealt with the relationship between sets and division at the Fourth grade level. It was a one page passage consisting of two simple examples of how division may be interpreted in terms of sets, followed by a set of exercises quite similar to the examples. The CMMT analysis of focus passage FP1 is given in Table 2.

Table 2

CMMT Analysis of Focus Passage FP1.

List.

		Dim. 1	Dim. 2		Weight
	Heading	10	1		1
		0	0		3
Exposition	Intro. Statement	4	1		2
		0	0		2
	Example followed by definition.	3	1	Word, illustration, symbol sequence.	4
		3	3		4
		3	2		1
		0	0		2
		1	1		2
		0	0		1
	Example followed by definition.	3	1	Word, illustration, symbol sequence.	4
		3	3		4
		3	2		1
		0	0		2
		1	1		2
		0	0		5
	Heading	10	1		1
		0	0		7

Table 2. cont.

Exercises	Exercise	7	1	Word,	7
		7	2	symbol.	1
		0	0		6
	Exercise	7	1	Word,	5
		7	2	symbol.	1
		0	0		6
	Directions	5	1		2
		0	0		
	Exercise	7	3	Illustration,	5
		7	2	symbol.	1
		0	0		
	Exercise	7	3	Illustration,	5
		7	2	symbol.	1
	Directions	5	1		2
	Exercise	7	1	Word,	2
		7	2	symbol.	1
		0	0		1
	Exercise	7	1	Word,	2
		7	2	symbol.	1
		0	0		1

Dimension 1 interaction matrix.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
	1	2	0	1	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0	0
	3	2	0	16	0	0	0	0	0	0	0
	4	0	0	1	1	0	0	0	0	0	0
Dimension 1	5	0	0	0	0	2	0	2	0	0	0
Categories	6	0	0	0	0	0	0	0	0	0	0
	7	0	0	0	0	2	0	35	0	0	0
	8	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	0	1	0	0	1	0	0	0

A

B

Table 2. cont.

Dimension 2 interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	29	4	3	0	0	0	0
	2	6	0	2	0	0	0	0
Dimension 2	3	0	5	18	0	0	0	0
Categories	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Between dimension interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	4	0	0	0	0	0	0
	2	0	0	0	0	0	0	0
	3	8	2	8	0	0	0	0
	4	2	0	0	0	0	0	0
Dimension 1	5	4	0	0	0	0	0	0
Categories	6	0	0	0	0	0	0	0
	7	16	7	15	0	0	0	0
	8	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	2	0	0	0	0	0	0

Table 2. cont.

CMMT proportions.

Proportion of blank space in total passage	.41

Proportion of each category in non-blank material.

Category	Dimension 1	Dimension 2
1	.06	.54
2	.00	.12
3	.26	.34
4	.03	.00
5	.03	.00
6	.00	.00
7	.55	.00
8	.00	—
9	.00	—
10	.03	—

Proportion of exposition in non-blank material	.36
Proportion of content development in non-blank material	.36
Proportion of exercises in content development	.00
Proportion of exposition in content development	1.00
Proportion requiring responses in non-blank material	.61
Proportion of mathematics illustration in non-blank material	.34
Proportion of illustration in non-blank material	.34

The CMMT list for passage FP1 revealed that the passage consisted of two example-definition sequences followed by a set of exercises. The interaction matrices showed that there was some interaction among categories on dimension 1 (10 off-diagonal entries) while there was relatively more interaction between categories 1, 2, and 3 of dimension 2 (20 off-diagonal entries). The most numerous messages were examples (3) and exercises (7) and these two types of messages were represented in all of the three basic modes (rows 3 and 7 of the between-dimension matrix). The CMMT proportions indicated that the content development was entirely expository and accounted for a little more than one-third of the passage while the exercises accounted for a little less than two-thirds of the passage.

Focus Passage FP3. This fifth grade passage concerned the Commutative Property of addition. It consisted of a short sequence of developmental questions culminating in a statement of the Commutative Property. This developmental material was followed by a set of simple exercises. The CMMT analysis of focus passage FP3 is given in Table 3.

Table 3

CMMT Analysis of Focus Passage FP3.

List.

		Dim.1	Dim.2		Weight
Heading		10	1		4
		0	0		4
Developmental Exercises		6	1		6
		0	0		2
		6	5	Representational Illustration	24
		6	1	Symbol Word Combinations	2
		6	2		1
		6	1		3
		0	0		2
		6	2		3
		0	0		1
	Directions	5	1		1
		0	0		3
		6	2		8
Generalization		2	1		10
		0	0		2
Developmental Exercises		6	1		6
		6	2		2
		0	0		8
Heading		10	1		1
Exercises	Directions	5	1		1
		0	0		3
		7	2		8
	Directions	5	1		4
		7	2		8

Table 3. cont.

Dimension 1 interaction matrix.

Dimension 1 Categories

		1	2	3	4	5	6	7	8	9	10
	1	0	0	0	0	0	0	0	0	0	0
	2	0	9	0	0	0	1	0	0	0	0
	3	0	0	0	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0	0	0	0
Dimension 1	5	0	0	0	0	3	1	2	0	0	0
Categories	6	0	1	0	0	1	52	0	0	0	1
	7	0	0	0	0	1	0	14	0	0	0
	8	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	0	0	1	1	0	0	0	3

Dimension 2 interaction matrix.

Dimension 2 Categories

		1	2	3	4	5	6	7
	1	31	6	0	0	1	0	0
	2	5	24	0	0	0	0	0
Dimension 2	3	0	0	0	0	0	0	0
Categories	4	0	0	0	0	0	0	0
	5	1	0	0	0	23	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Table 3. cont.

Between dimension interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
Dimension 1 Categories	1	0	0	0	0	0	0	0
	2	10	0	0	0	0	0	0
				A				
	3	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0
	5	6	0	0	0	0	0	0
	6	17	14	0	0	24	0	0
				B				
	7	0	16	0	0	0	0	0
	8	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	5	0	0	0	0	0	0

CMMT Proportions.

Proportion of blank space in total passage	.21

Proportion of each category in non-blank material.

Category	Dimension 1	Dimension 2
1	.00	.42
2	.11	.32
3	.00	.00
4	.00	.00
5	.07	.26
6	.60	.00
7	.16	.00
8	.00	—
9	.00	—
10	.05	—

Proportion of exposition in non-blank material	.11
Proportion of content development in non-blank material	.71
Proportion of exercises in content development	.85
Proportion of exposition in content development	.15

Table 3. cont.

Proportion requiring responses in non-blank material	.83
Proportion of mathematics illustration in non-blank material	.26
Proportion of illustration in non-blank material	.26

The CMMT list for passage FP3 indicated that the basic sequencing of this passage consisted of developmental exercises, generalization, developmental exercises, and comprehension level exercises. The matrix for dimension 1 revealed a number of interactions (10 off-diagonal entries) all involving categories 5, 6, and 7. While there was little interaction between illustration and written material shown in the dimension 2 matrix, there were a number of interactions between word and symbol entries (11 off-diagonal entries in region A). The between dimension matrix showed that more than one representational mode was used only in the case of developmental exercises (row 6). The CMMT proportions demonstrated that the passage consisted of about 15 percent exposition with about 83 percent of the messages requiring responses. While about 26 percent of the passage was in the illustrative mode this consisted of a single illustration.

Focus Passage FP5. In this junior high school level passage multiplication for rational numbers was defined. The passage consisted of a series of developmental exercises interwoven with exposition. Most of the passage was developmental with only a few practice exercises included at the end. The CMMT analysis of the passage is given in Table 4.

Table 4

CMMT Analysis of Focus Passage FP5.

		List.			
		Dim. 1	Dim. 2		Weight
		0	0		12
	Heading	10	1	Words	3
		0	0		2
General Explanation and Developmental Exercises		4	1		14
	Directions	5	1		1
		6	1		5
		4	1		2
		6	1		2
	Directions	5	1		1
		4	1		6
		6	1		4
		4	1		6
		0	0		1
		4	1		4
		6	1		2
	Definition	1	1		10
	Explanation	4	1		6
		0	0		1
	Example	3	2	Symbols	60
		0	0		2
	Definition	1	1	Words	5
		1	2	Symbols	2
		0	0		5
General Explanation and Developmental Exercises	Directions	5	1	Words	1
		6	1		5
	Directions	5	1		2
		4	1		5
		6	1		5
		0	0		1
	Definition	1	1	Word Symbol Combinations	7
		1	2		1
Developmental Exercises	Directions	5	1		1
		6	1		2
		6	2		1
		6	1		4
		6	1		2
		6	2		1
		6	1		5
	Directions	5	1		5
		0	0		1
Exercises		7	2		4

Table 4. Cont.

Dimension 1 interaction matrix.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
	1	22	0	0	1	2	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0	0
	3	1	0	59	0	0	0	0	0	0	0
	4	0	0	1	37	1	4	0	0	0	0
Dimension 1	5	0	0	0	2	5	3	1	0	0	0
Categories	6	2	0	0	2	3	31	0	0	0	0
	7	0	0	0	0	0	0	3	0	0	0
	8	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	0	1	0	0	0	0	0	2

Dimension 2 interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	109	6	0	0	0	0	0
	2	5	63	0	0	0	0	0
Dimension 2	3	0	0	0	0	0	0	0
Categories	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Table 4. cont.

Between dimension interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	22	3	0	0	0	0	0
	2	0	0	0	0	0	0	0
				A				
	3	0	60	0	0	0	0	0
	4	43	0	0	0	0	0	0
Dimension 1 Categories	5	11	0	0	0	0	0	0
	6	36	2	0	0	0	0	0
				B				
	7	0	4	0	0	0	0	0
	8	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	3	0	0	0	0	0	0

CMMT Proportions.

Proportion of blank space in total passage	.12

Proportion of each category in non-blank material.

Category	Dimension 1	Dimension 2
1	.14	.63
2	.00	.37
3	.33	.00
4	.23	.00
5	.06	.00
6	.21	.00
7	.02	.00
8	.00	—
9	.00	—
10	.02	—

Proportion of exposition in non-blank material	.70
Proportion of content development in non-blank material	.91
Proportion of exercises in content development	.23
Proportion of exposition in content development	.77

Table 4. cont.

Proportion requiring responses in non-blank material	.28
Proportion of mathematics illustration in non-blank material	.00
Proportion of illustration in non-blank material	.00

The CMMT list for passage FP5 showed a varied sequence of developmental exercises interwoven with general explanation, examples, and definitions. The dimension 1 matrix showed a number of interactions among seven categories of the system (24 off-diagonal entries). There were 13 interactions between exposition and messages requiring responses in this passage (13 entries in the shaded region). The dimension 2 matrix revealed interactions only between words and symbols (11 entries in region A). The between dimension matrix showed that except for the examples which were entirely symbols the passage was predominantly words. The CMMT proportions showed that the passage consisted of 90 percent content development of which about 80 percent was exposition. About 30 percent of the messages required responses. There were no illustrations.

The CMMT analysis of passage FP5 may be somewhat misleading in one respect. In the passage the examples were physically isolated from the rest of the passage as shown in the analysis. However, the reader was referred to the various examples throughout the developmental material. Thus, there was really more interaction between examples and other categories than indicated by the analysis.

Focus Passage FP8. This passage concerned the use of grouping symbols in beginning algebra. The passage consisted of an expository section, in which general explanation and examples were interwoven, followed by numerous exercises on removing parentheses from expressions. The CMMT analysis for the passage is given in Table 5.

Table 5

CMMT Analysis of Focus Passage FP8.

List.

		Dim. 1	Dim. 2		Weight
Exposition	Heading	10	1		2
		0	0		6
	Explanation	4	1	Words	6
	Example	3	2	Symbols	8
	Explanation	4	1	Words	11
	Heading	10	1		1
		0	0		7
	Example	3	1	Word, Symbol	3
		3	2		15
		0	0		5
	Explanation	4	1	Words	18
		0	0		2
	Example	3	2	Word, Symbol	7
		3	1		6
		0	0		8
Exercises	Heading	10	1		1
		0	0		7
	Directions	5	1	Words	4
	Practice	7	2	Symbols	12
		0	0		12
		7	2		18
		0	0		6

Table 5. cont.

Dimension 1 interaction matrix.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
	1	0	0	0	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0	0
	3	0	0	36	2	0	0	0	0	0	1
	4	0	0	2	32	0	0	0	0	0	1
Dimension 1	5	0	0	0	0	3	0	1	0	0	0
Categories	6	0	0	0	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	29	0	0	0
	8	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	1	1	1	0	0	0	0	1

Dimension 2 interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	48	4	0	0	0	0	0
	2	3	56	0	0	0	0	0
Dimension 2	3	0	0	0	0	0	0	0
Categories	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Table 5. cont.

Between dimension interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0
				A				
	3	9	30	0	0	0	0	0
	4	35	0	0	0	0	0	0
	5	4	0	0	0	0	0	0
Dimension 1	6	0	0	0	0	0	0	0
Categories				B				
	7	0	30	0	0	0	0	0
	8	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	4	0	0	0	0	0	0

CMMT Proportions.

Proportion of blank space in total passage .41

Proportion of each category in non-blank material.

Category	Dimension 1	Dimension 2
1	.00	.47
2	.00	.53
3	.35	.00
4	.31	.00
5	.04	.00
6	.00	.00
7	.26	.00
8	.00	—
9	.00	—
10	.04	—

Proportion of exposition in non-blank material	.67
Proportion of content development in non-blank material	.67
Proportion of exercises in content development	.00
Proportion of exposition in content development	1.00
Proportion requiring responses in non-blank material	.30
Proportion of mathematics illustration in non-blank material	.00
Proportion of illustration in non-blank material	.00

The list for passage FP8 revealed an exposition-followed-by-exercises sequence. The exposition consisted of general explanation and examples. The dimension 1 matrix shows the passage contained little interaction between mathematical messages (only 5 off-diagonal entries for categories 1 - 9). The dimension 2 matrix showed some interaction between words and symbols (7 off-diagonal entries in region A). The between dimension matrix showed only four of the nine mathematical content categories were present with the explanation consisting entirely of words and the exercises consisting entirely of symbols. The CMMT proportions indicated about two-thirds of the passage was exposition and about one-third was comprehension level exercises. The passage was about 47 percent words and about 53 percent symbols with no illustration.

Focus Passage FP11. This passage from second year algebra developed the definition of fractional exponents. The passage began with general and specific explanation leading to the definition. This was followed by some practice exercises with answers which served to complete the development of the concept of fractional exponents. The passage concluded with numerous exercises dealing with changing back and forth between fractional exponent and radical notation and with evaluating numerical expressions containing fractional exponents. The CMMT analysis for the passage is given in Table 6.

Table 6

CMMT Analysis of Focus Passage FP11.

List.

		Dim. 1	Dim. 2		Weight
	Heading	10	1	Word Symbol Combinations	2
Exposition	Examples	3	1		9
		3	2		8
		0	0		8
		3	1		9
		0	0		3
		3	2		10
		0	0		10
	Definition	1	1		3
		1	2		1
	Explanation	4	1		8
	Heading	10	1		1
Developmental Exercises		0	0		3
	Directions	5	1		4
		6	2		3
		0	0		1
	Directions	5	1		2
		6	2		3
		0	0		3
	Heading	10	1		1
		0	0		3
	Specific Explanation	3	2		8
		0	0		4
	Heading	10	1		3
Exercises	Directions	5	1		2
		0	0		2
	Practice	7	2		14
		0	0		2
	Directions	5	1		2
		0	0		2
	Practice	7	2		14
		0	0		2
	Directions	5	1		2
		0	0		2
	Practice	7	2		16
	Directions	5	1		2
		0	0		2
	Practice	7	2		9
		0	0		3
	Directions	5	1		1
		0	0		3
	Practice	7	2		3
		0	0		1

Table 6. cont.

Dimension 1 interaction matrix.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
Dimension 1 Categories	1	3	0	0	1	0	0	0	0	0	0
	2	0	0	0	0	0	0	0	0	0	0
	3	1	0	42	0	0	0	0	0	0	1
	4	0	0	0	7	0	0	0	0	0	1
	5	0	0	0	0	8	2	5	0	0	0
	6	0	0	0	0	1	4	0	0	0	1
	7	0	0	0	0	4	0	51	0	0	0
	8	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	2	0	2	0	0	0	0	3

Dimension 2 interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
Dimension 2 Categories	1	40	11	0	0	0	0	0
	2	10	78	0	0	0	0	0
	3	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Table 6. cont.

Between dimension interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	3	1	0	0	0	0	0
	2	0	0	0	0	0	0	0
				A				
	3	18	26	0	0	0	0	0
	4	8	0	0	0	0	0	0
Dimension 1	5	15	0	0	0	0	0	0
Categories	6	0	6	0	0	0	0	0
				B				
	7	0	56	0	0	0	0	0
	8	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	7	0	0	0	0	0	0

CMMT Proportions.

Proportion of blank space in total passage .28

Proportion of each category in non-blank material

Category	Dimension 1	Dimension 2
1	.03	.37
2	.00	.63
3	.32	.00
4	.06	.00
5	.11	.00
6	.04	.00
7	.40	.00
8	.00	—
9	.00	—
10	.05	—

Proportion of exposition in non-blank material	.40
Proportion of content development in non-blank material	.45
Proportion of exercises in content development	.10
Proportion of exposition in content development	.90
Proportion requiring responses in non-blank material	.55
Proportion of mathematics illustration in non-blank material	.00
Proportion of illustration in non-blank material	.00

The CMMT list for passage FP11 reflected a sequence consisting of exposition, developmental exercises, and comprehension level exercises. The dimension 1 interaction matrix showed there was some interaction among categories (21 off-diagonal entries) of which twelve were interactions between categories 5, 6, and 7 (12 off-diagonal entries in region B). The dimension 2 matrix showed some interaction between words and symbols (21 off-diagonal entries). The between dimension matrix showed that only in the case of examples were both words and symbols used (row 3). The CMMT proportions indicated that the messages were about 45 percent content development of which about 90 percent was exposition and about 10 percent developmental exercises. About 54 percent of the messages required responses. About 37 percent of the messages were words and 63 percent were symbols. There were no illustrations.

Focus Passage FP12. This passage from trigonometry concerned the development of functions relating the time a particle moves along a square at a constant rate to the first and second coordinates of the particles. The passage consisted of exposition followed by exercises and applications demanding modifications of the developed functions. The CMMT analysis of the passage is given in Table 7.

Table 7

CMMT Analysis of Focus Passage FP12.

List.

		Dim.1	Dim.2		Weight
	Heading	10	1		2
Exposition		0	0		2
	Explanation	4	1	Words	28
	Examples	3	1	Words, Illustration, Symbols sequence	1
		3	4		12
		3	2		4
		3	1		1
		3	4		12
		3	2		4
		3	1		1
		3	4		12
		3	2		4
	Explanation	4	1	Words	19
		0	0		1
	Heading	10	1		1
		0	0		7
Exercises and Problems	Directions	5	1	Words	1
		0	0		3
	Exercises	7	2	Symbols	6
		0	0		2
		7	1	Words	4
		0	0		4
	Applications	8	1	Words	33
		8	4	Illustration	22
		0	0		1
		8	1	Words	6
		0	0		2

Table 7. cont.

Dimension 1 interaction matrix.

Dimension 1 Categories

		1	2	3	4	5	6	7	8	9	10
	1	0	0	0	0	0	0	0	0	0	0
				A							
	2	0	0	0	0	0	0	0	0	0	0
	3	0	0	50	1	0	0	0	0	0	0
	4	0	0	1	45	0	0	0	0	0	1
Dimension 1	5	0	0	0	0	0	0	1	0	0	0
Categories	6	0	0	0	0	0	0	0	0	0	0
									B		
	7	0	0	0	0	0	0	9	1	0	0
	8	0	0	0	0	0	0	0	60	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	0	1	1	0	0	0	0	0

Dimension 2 interaction matrix.

Dimension 2 Categories

		1	2	3	4	5	6	7
	1	91	A 1	0	4	0	0	0
	2	4	14	0	0	0	0	0
Dimension 2	3	0	0	0	0	0	0	0
					B			
Categories	4	1	3	0	54	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Table 7. cont.

Between dimension interaction matrix.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0
					A			
	3	0	12	0	36	0	0	0
	4	47	0	0	0	0	0	0
Dimension 1 Categories	5	1	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
					B			
	7	4	6	0	0	0	0	0
	8	39	0	0	22	0	0	0
	9	0	0	0	0	0	0	0
	10	6	0	0	0	0	0	0

CMMT Proportions.

Proportion of blank space in total passage .11

Proportion of each category in non-blank material

Category	Dimension 1	Dimension 2
1	.00	.56
2	.00	.11
3	.28	.00
4	.27	.34
5	.01	.00
6	.00	.00
7	.06	.00
8	.35	—
9	.00	—
10	.03	—

Proportion of exposition in non-blank material	.56
Proportion of content development in non-blank material	.56
Proportion of exercises in content development	.00
Proportion of exposition in content development	1.00
Proportion requiring responses in non-blank material	.41
Proportion of mathematics illustration in non-blank material	.34
Proportion of illustration in non-blank material	.34

The CMMT list for passage FP12 demonstrated an exposition-followed-by-exercises sequencing. The exposition consisted of general explanation which was followed by some examples which were followed by more explanation. The exercises consisted of comprehension level exercises followed by some application level exercises. The dimension 1 interaction matrix showed that there were few interactions (only 7 off-diagonal interactions). There were relatively more interactions on dimension 2 (13 off-diagonal entries) with eight interactions between written material and illustration (8 entries in the shaded regions). The between dimension interaction matrix showed that the examples consisted of symbols and illustration (row 3) while the general explanation consisted of words. Exercises and problems were presented in all three representational modes. The CMMT proportions demonstrated that the passage was about 55 percent exposition and 41 percent exercises. The representation was 56 percent words, 10 percent symbols, and 34 percent illustration.

Descriptive Results for the Contrived Passages

To illustrate how CMMT analysis can be used to compare and contrast text passages, six pairs of contrived passages were written following contrasting organizational plans. These passages are included in Appendix D. In this section, a description of each pair of contrived passages is given. Each of these descriptions is followed by the CMMT analyses of the passages in the pair described. The manner in which these CMMT analyses reflect the planned organization of the passages in the pair is discussed.

Contrived Passages CP2a and CP2b. These contrived passages were based on focus passage FP2, which was a grade five treatment of circumference. Contrived passage CP2a was constructed to contain a high proportion of developmental exercises and a low proportion of explanation. Exercises were integrated with expository material. Contrived passage CP2b was constructed to contain a high proportion of explanation, no developmental exercises, and separation of exposition from exercises. An attempt was made to hold dimension 2 constant over both passages. The CMMT analyses of these passages are given in Table 8.

Table 8

CMMT Analyses of Contrived Passages CP2a and CP2b.

List for CP2a.

	Dim. 1	Dim. 2	Weight
Heading	10	1	1
	0	0	3
Exposition and Developmental Exercises	6	1	3
	0	0	3
	6	3	5
	0	0	1
	1	1	3
	6	1	15
	0	0	2
	0	0	1
	6	5	3
	0	0	2
	6	5	3
	0	0	2
	6	5	3
	0	0	6
	5	1	7
	0	0	7
	6	7	19
	0	0	3
	6	1	14
	0	0	2
	2	1	9
	1	1	4
	1	2	1
	6	1	4
	0	0	2
	6	1	5
	6	2	4
	0	0	11
	6	1	13
	0	0	3
Heading	10	1	1
	0	0	3
Exercises and Problems	5	1	3
	0	0	1
	7	3	3
	0	0	3
	7	3	3
	0	0	3
	7	2	1
	0	0	1
	7	2	1
	0	0	1
	8	1	4
	0	0	4
	8	3	4
	0	0	4
	8	1	7
	0	0	1
	8	1	9
	0	0	3
	8	1	11
	0	0	6
	8	5	6
	0	0	5
	9	1	2
	0	0	2
	9	1	9
	0	0	5
	9	3	4
	0	0	4
	9	1	12
	8	2	2
	8	1	6

List for CP2b.

	Dim. 1	Dim. 2	Weight
Heading	10	1	1
	0	0	3
Exposition	4	1	3
	0	0	3
	4	3	5
	0	0	1
	1	1	3
	4	1	11
	3	1	6
	0	0	1
	3	5	3
	0	0	2
	3	5	3
	0	0	2
	3	5	3
	0	0	6
	3	1	9
	0	0	8
	3	7	13
	0	0	6
	3	1	15
	0	0	1
	2	1	9
	1	1	4
	1	2	1
	3	1	5
	0	0	1
	2	1	5
	2	2	4
	0	0	11
	4	1	13
	0	0	3
Heading	10	1	1
Exercises and Problems	5	1	2
	0	0	12
	7	7	18
	0	0	7
	5	1	3
	0	0	1
	7	3	3
	0	0	3
	7	3	3
	0	0	3
	7	3	3
	0	0	3
	7	3	3
	0	0	3
	7	2	1
	0	0	1
	7	2	1
	0	0	1
	7	2	1
	0	0	1
	7	2	1
	0	0	1
	8	1	4
	0	0	4
	8	3	4
	0	0	4
	8	1	7
	0	0	1
	8	1	6
	0	0	2
	8	1	11
	0	0	6
	8	5	7
	0	0	8

Table 8. cont.

Dimension 1 interaction matrix for CP2a.

Dimension 1 Categories

Dimension 1 Categories	1	2	3	4	5	6	7	8	9	10
1	6	0	0	0	0	2	0	0	0	0
2	1	8	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	8	1	1	0	0	0
6	1	1	0	0	1	87	0	0	0	1
7	0	0	0	0	0	0	7	1	0	0
8	0	0	0	0	0	0	0	47	1	0
9	0	0	0	0	0	0	0	1	26	0
10	0	0	0	0	1	1	0	0	0	0

Dimension 2 interaction matrix fo- CP2a.

Dimension 2 Categories

Dimension 2 Categories	1	2	3	4	5	6	7
1	131	3	4	0	2	0	1
2	4	6	0	0	0	0	0
3	3	1	15	0	0	0	0
4	0	0	0	0	0	0	0
5	2	0	0	0	3	0	0
6	0	0	0	0	0	0	0
7	1	0	0	0	0	0	18

Dimension 1 interaction matrix for CP2b.

Dimension 1 Categories

Dimension 1 Categories	1	2	3	4	5	6	7	8	9	10
1	6	0	1	1	0	0	0	0	0	0
2	1	16	0	1	0	0	0	0	0	0
3	0	2	55	0	0	0	0	0	0	0
4	1	0	1	29	0	0	0	0	0	1
5	0	0	0	0	3	0	2	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	1	0	32	1	0	0
8	0	0	0	0	0	0	0	38	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	1	1	0	0	0	0	0

Dimension 2 interaction matrix for CP2b.

Dimension 2 Categories

Dimension 2 Categories	1	2	3	4	5	6	7
1	100	2	3	0	2	0	2
2	3	6	0	0	0	0	0
3	2	1	18	0	0	0	0
4	0	0	0	0	0	0	0
5	1	0	0	0	14	0	0
6	0	0	0	0	0	0	0
7	2	0	0	0	0	0	29

Table 8. cont.

Between Dimension interaction matrix for CP2a.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
Dimension 1 Categories	1	7	1	0	0	0	0	0
	2	9	0	0	0	0	0	0
					A			
	3	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0
	5	10	0	0	0	0	0	0
	6	54	4	5	0	9	0	19
					B			
	7	0	2	6	0	0	0	0
	8	37	2	4	0	6	0	0
	9	23	0	4	0	0	0	0
	10	2	0	0	0	0	0	0

Between Dimension interaction matrix for CP2a.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
Dimension 1 Categories	1	7	1	0	0	0	0	0
	2	14	4	0	0	0	0	0
					A			
	3	35	0	0	0	9	0	13
	4	27	0	5	0	0	0	0
	5	5	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
					B			
	7	0	4	12	0	0	0	18
	8	28	0	4	0	7	0	0
	9	0	0	0	0	0	0	0
	10	2	0	0	0	0	0	0

CMMT Proportions		Passage CP2a		Passage CP2b	
Proportion of blank space in total passage		.32		.36	
Proportion of each category in non-blank material.		Dim. 1	Dim. 2	Dim. 1	Dim. 2
Category:	1	.04	.69	.04	.61
	2	.04	.04	.09	.05
	3	.00	.09	.29	.11
	4	.00	.00	.16	.00
	5	.05	.07	.03	.08
	6	.45	.00	.00	.00
	7	.09	.09	.18	.16
	8	.24	—	.20	—
	9	.13	—	.00	—
	10	.01	—	.01	—
Proportion of exposition in non-blank material		.08		.59	
Proportion of content development in non-blank material		.53		.59	
Proportion of exercises in content development		.84		.00	
Proportion of exposition in content development		.16		1.00	
Proportion requiring responses in non-blank material		.91		.40	
Proportion of mathematics illustration in non-blank material		.17		.19	
Proportion of illustration in non-blank material		.26		.34	

The organizations of contrived passages CP2a and CP2b were reflected by the CMMT analyses of the passages. The CMMT lists indicated that essentially the same sequence was used for both passages except that developmental exercises were used in passage CP2a where explanation was used in passage CP2b. The lists also revealed that similar sequences of dimension 2 categories are found in both passages.

The dimension 1 interaction matrices revealed that passage CP2a contained seven interactions between exposition and exercises and only one interaction between exposition categories, while passage CP2b contained one interaction between exposition and exercises and eight interactions among exposition categories. These interaction differences reflected the planned degree of integration of exposition and exercises in the two passages. The dimension 2 interaction matrices showed a high degree of similarity between the two passages in the interaction of categories in the representational mode as planned.

The CMMT proportions reflected the planned differences in quantity of explanation and exercises between the two passages. Passage CP2a contained less than 10 percent exposition while passage CP2b contained about 60 percent exposition. About 90 percent of the messages in passage CP2a required responses while only about 40 percent of passage CP2b required responses. The dimension 2 proportions were approximately the same for both passages.

Contrived Passages CP4a and CP4b. These passages were based on focus passage FP4 which was a sixth grade passage on the use of equations and inequalities to solve verbal problems. Contrived passage CP4a was constructed to contain a high proportion of examples and a low proportion of general explanation while contrived passage CP4b was constructed to contain no examples and a high proportion of general explanation. Passage CP4a utilized non-mathematical motivational illustrations and passage CP4b utilized illustrations with mathematical content. The CMMT analyses of these passages are given in Table 9.

Table 9

CMMT Analyses of Contrived Passages CP4a and CP4b.

List for CP4a.

	Dim. 1	Dim. 2		Weight
Heading	10	1		2
	0	0		2
Examples	3	1	Words and Symbols	7
	0	0		2
	3	2		1
	0	0		2
	3	1		1
	3	2		1
	3	1		3
	0	0		3
	3	1		8
	0	0		1
	3	2		1
	0	0		2
	3	1		4
	3	2		1
	3	1		4
	0	0		3
Heading	10	1		1
	0	0		3
Directions	5	1		4
Applications and Motivational Illustrations	8	1		9
	0	0		3
	10	6		12
	8	1		10
	0	0		5
	10	6		11
	0	0		6
	8	1		5
	0	0		5
	10	6		12
	0	0		6
	8	1		11
	0	0		5
	10	6		10
	0	0		6
	8	1		2
	8	2		1
	8	1		1
	8	2		1
	8	1		2
	0	0		6
	10	6		12
	0	0		7

List for CP4b.

	Dim. 1	Dim. 2		Weight
Heading	10	1		2
	0	0		2
General Explanation	4	1	Words and Symbols	7
	0	0		1
	4	1		6
	0	0		2
	4	1		4
	4	2		1
	0	0		3
	4	1		2
	4	2		1
	4	1		2
	4	2		1
	0	0		6
	4	1		6
	0	0		2
Heading	10	1		1
	0	0		3
Directions	5	1		4
Applications and Mathematical Illustrations	8	1		9
	0	0		3
	8	5		12
	8	1		10
	0	0		5
	8	5		11
	0	0		6
	8	1		5
	0	0		5
	8	5		12
	0	0		6
	8	1		11
	0	0		5
	8	5		10
	0	0		6
	8	1		2
	8	2		1
	8	1		1
	8	2		1
	8	1		2
	0	0		6
	8	7		12
	0	0		7

Table 9. cont.

Dimension 1 interaction matrix for CP4a.

Dimension 1 Categories

Dimension 1 Categories	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	30	0	0	0	0	0	0	1
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	3	0	0	1	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	37	0	5
9	0	0	0	0	0	0	0	0	0	0
10	0	0	1	0	1	0	0	4	0	53

Dimension 2 interaction matrix for CP4a.

Dimension 2 Categories

Dimension 2 Categories	1	2	3	4	5	6	7
1	63	6	0	0	0	5	0
2	6	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	4	0	0	0	0	92	0
7	0	0	0	0	0	0	0

Dimension 1 interaction matrix for CP4b.

Dimension 1 Categories

Dimension 1 Categories	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	29	0	0	0	0	0	1
5	0	0	0	0	3	0	0	1	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	98	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	1	1	0	0	0	0	1

Dimension 2 interaction matrix for CP4b.

Dimension 2 Categories

Dimension 2 Categories	1	2	3	4	5	6	7
1	64	5	0	0	4	0	1
2	5	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	4	0	0	0	41	0	0
6	0	0	0	0	0	0	0
7	0	0	0	0	0	0	11

Table 9. cont.

Between Dimension interaction matrix for CP4a.

Dimension 2 Categories

		1	2	3	4	5	6	7
	1	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0
					A			
	3	27	4	0	0	0	0	0
	4	0	0	0	0	0	0	0
Dimension 1 Categories	5	4	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
					B			
	7	0	0	0	0	0	0	0
	8	40	2	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	3	0	0	0	0	57	0

Between Dimension interaction matrix for CP4b.

Dimension 2 Categories

		1	2	3	4	5	6	7
	1	0	0	0	0	0	0	0
	2	0	0	0	0	0	0	0
					A			
	3	0	0	0	0	0	0	0
	4	27	3	0	0	0	0	0
Dimension 1 Categories	5	4	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
					B			
	7	0	0	0	0	0	0	0
	8	40	2	0	0	45	0	12
	9	0	0	0	0	0	0	0
	10	3	0	0	0	0	0	0

CMMT Proportions		Passage CP4a		Passage CP4b	
Proportion of blank space in total passage		.33		.33	
Proportion of each category in non-blank material.		Dim. 1	Dim. 2	Dim. 1	Dim. 2
Category:	1	.00	.54	.00	.55
	2	.00	.04	.00	.04
	3	.23	.00	.00	.00
	4	.00	.00	.22	.00
	5	.03	.00	.03	.33
	6	.00	.41	.00	.00
	7	.00	.00	.00	.08
	8	.31	—	.73	—
	9	.00	—	.00	—
	10	.43	—	.02	—
Proportion of exposition in non-blank material		.23		.22	
Proportion of content development in non-blank material		.23		.22	
Proportion of exercises in content development		.00		.00	
Proportion of exposition in content development		1.00		1.00	
Proportion requiring responses in non-blank material		.34		.76	
Proportion of mathematics illustration in non-blank material		.00		.42	
Proportion of illustration in non-blank material		.41		.42	

The CMMT lists for contrived passages CP4a and CP4b revealed the similar sequential nature of the two passages. These lists were almost identical with dimension 1 categories 3 and 10 and dimension 2 category 6 in passage CP4a replaced by dimension 1 categories 4 and 8 and dimension 2 categories 5 and 7 in passage CP4b. The interaction matrices revealed little difference between the two passages on either dimension with the exception of nine interactions between the exercises and motivational illustrations on dimension 1 in passage CP4a.

The major differences between these passages were reflected by the CMMT proportions. Passage CP4a contained about 22 percent examples and no general explanation, while passage CP4b contained no examples and about 22 percent general explanation. The dimension 1 proportions also showed that in passage CP4a about 33 percent of the messages required responses, while in passage CP4b about 75 percent of the messages required responses. This difference was a result of the fact that the illustrations were of a non-mathematical nature in passage CP4a, but contained mathematical information which could be used in the solution of the problems in passage CP4b. The dimension 2 proportions revealed that word and symbol content were nearly the same for both passages. Both passages consisted of about 40 percent illustration, but in passage CP4a these illustrations were motivational while in passage CP4b 33 percent were illustrations of physical situations and about 8 percent were tables.

Contrived Passages CP6a and CP6b. These passages were based on focus passage FP6 which was a junior high treatment of the probability of the event A or B where events A and B have given probabilities. Dimension 1 categories were held constant in the construction of these passages. Passage CP6a was constructed to contain high proportions of symbols and illustrations and low proportions of words. Passage CP6b was constructed to contain high proportions of words and few symbols or illustrations. The CMMT analyses for these contrived passages is given in Table 10.

Table 10

CMMT Analyses of Contrived Passages CP6a and CP6b.

List for CP6a.

	Dim. 1	Dim. 2		Weight
Heading	10	1	Words	1
	0	0		3
Example	3	1		2
	0	0		4
	3	5	Illustration	5
	0	0		2
	3	2	Words, Symbols	5
	0	0		6
	3	2		1
Question	6	2		1
	0	0		6
Example	3	1		4
	3	2		3
	3	1		3
	3	2		2
	0	0		2
	3	1		8
Definition	1	1		6
	0	0		2
Example	3	1		4
	3	5	Illustration	5
	0	0		8
	3	2	Words, Symbols	2
	0	0		5
Question	6	2		1
	0	0		3
Example	3	1		5
	3	2		3
	3	1		1
	3	2		2
	0	0		2
	3	1		8
Explanation	4	1		5
	0	0		3
Generalization	2	1		3
	2	2		2
	0	0		3
Explanation	4	1		5
	0	0		3
Generalization	2	2		1
	2	1		3
Definition	1	2		1
	1	1		2
	1	2		1
Generalization	2	1		1
	2	2		3
Heading	10	1		1
	0	0		3
Applications	8	5	Illustration	16
	8	2	Words, Symbols	8
	0	0		8
	8	1		2
	8	2		5
	8	5	Illustration	4
	0	0		14
	8	1	Words, Symbols	2
	0	0		2
	8	2		1
	3	1		3
	8	2		3
	8	1		8
	0	0		19
	8	5	Illustration	14
	0	0		4
	8	2	Words, Symbols	10
	0	0		10
	8	1		4

List for CP6b.

	Dim. 1	Dim. 2		Weight
Heading	10	1	Words	1
	0	0		3
Example	3	1		12
	3	2	Symbols	1
Question	6	1	Words	3
Examples	3	1		22
	0	0		2
	3	1		9
Definition	1	1		6
	0	0		1
Example	3	1		48
Explanation	4	1		5
	0	0		3
Generalization	2	1		8
Explanation	4	1		5
	0	0		3
Generalization	2	1		3
Definition	1	1		4
Generalization	2	1		9
Heading	10	1		1
	0	0		3
Applications	8	1		24
	0	0		16
	8	1		13
	0	0		11
	8	1		11
	0	0		9
	8	1		6
	0	0		2
	8	1		10
	0	0		2
	8	1		15
	0	0		1
	8	1		3
	0	0		3
	8	1		3
	0	0		3
	8	1		3
	0	0		1
	8	1		3
	0	0		1
	8	1		3
	0	0		1
	8	1		4

Table 10. cont.

Dimension 1 interaction matrix for CP6a.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
Dimension 1 Categories	1	0	1	1	0	0	0	0	0	0	0
	2	1	10	0	1	0	0	0	0	0	1
	3	1	0	59	1	0	2	0	1	0	0
	4	0	2	0	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0	0	0	0
	6	0	0	2	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0	0	0	0
	8	0	0	1	0	0	0	0	75	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	1	0	0	0	0	1	0	0

Dimension 2 interaction matrix for CP6a.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
Dimension 2 Categories	1	62	12	0	0	4	0	0
	2	15	30	0	0	1	0	0
	3	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0
	5	1	4	0	0	39	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Dimension 1 interaction matrix for CP6b.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
Dimension 1 Categories	1	0	1	1	0	0	0	0	0	0	0
	2	1	17	0	1	0	0	0	0	0	1
	3	1	0	89	1	0	1	0	0	0	0
	4	0	2	0	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0	0	0	0
	6	0	0	1	0	0	2	0	0	0	0
	7	0	0	0	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0	97	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	1	0	0	0	0	1	0	0

Dimension 2 interaction matrix for CP6b.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
Dimension 2 Categories	1	232	1	0	0	0	0	0
	2	1	0	0	0	0	0	0
	3	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Table 10. cont.

Between Dimension interaction matrix for CP6a.

Dimension 1 Categories			Dimension 2 Categories						
			1	2	3	4	5	6	7
	A	1	8	2	0	0	0	0	0
		2	7	6	0	0	0	0	0
		3	36	18	0	0	10	0	0
		4	10	0	0	0	0	0	0
	B	5	0	0	0	0	0	0	0
		6	0	2	0	0	0	0	0
		7	0	0	0	0	0	0	0
		8	16	27	0	0	34	0	0
		9	0	0	0	0	0	0	0
		10	2	0	0	0	0	0	0

Between Dimension interaction matrix for CP6b.

Dimension 1 Categories			Dimension 2 Categories						
			1	2	3	4	5	6	7
	A	1	10	0	0	0	0	0	0
		2	20	0	0	0	0	0	0
		3	91	1	0	0	0	0	0
		4	10	0	0	0	0	0	0
	B	5	0	0	0	0	0	0	0
		6	3	0	0	0	0	0	0
		7	0	0	0	0	0	0	0
		8	98	0	0	0	0	0	0
		9	0	0	0	0	0	0	0
		10	2	0	0	0	0	0	0

CMMT Proportions		Passage CP6a		Passage CP6b	
Proportion of blank space in total passage		.39		.30	
Proportion of each category in non-blank material.		Dim. 1	Dim. 2	Dim. 1	Dim. 2
Category:	1	.06	.44	.04	.99
	2	.07	.31	.08	.01
	3	.36	.00	.39	.00
	4	.06	.00	.04	.00
	5	.00	.25	.00	.00
	6	.01	.00	.01	.00
	7	.00	.00	.00	.00
	8	.41	—	.41	—
	9	.00	—	.00	—
	10	.01	—	.01	—
Proportion of exposition in non-blank material		.55		.56	
Proportion of content development in non-blank material		.56		.58	
Proportion of exercises in content development		.02		.02	
Proportion of exposition in content development		.98		.98	
Proportion requiring responses in non-blank material		.44		.43	
Proportion of mathematics illustration in non-blank material		.25		.00	
Proportion of illustration in non-blank material		.25		.00	

The CMMT lists for contrived passages CP6a and CP6b demonstrated that the two passages followed almost identical sequences on dimension 1. The lists showed that passage CP6a consisted of combinations of words, symbols, and illustrations on dimension 2, while passage CP6b was almost entirely words.

The interaction matrices for the two passages supported the contention that the passages are similar sequentially on dimension 1. The larger entries on the diagonal of the dimension 1 matrix for passage CP6b indicated that the passage with high word representation took up more area than the passage in which symbols were used. The dimension 2 interaction matrices demonstrated the differences between the representational natures of the two passages. In passage CP6b there were only two interactions between words and symbols. The between dimension interaction matrices also showed these representational differences between the two passages.

The CMMT proportions for the two passages demonstrated that the passages are very similar on dimension 1. On dimension 2, passage CP6a was about 44 percent words, 31 percent symbols, and 25 percent illustration. Passage CP6b was over 99 percent words.

Contrived Passages CP7a and CP7b. In these passages an attempt was made to vary the dimension 1 sequence and hold all other aspects of the CMMT analyses constant. The passages were based on focus passage FP7 which was concerned with the development of some consequences of the axioms of affine geometry. In passage CP7a proofs preceeded statements of the theorems, resulting in a series of 4,2 sequences. In passage CP7b statements of the theorems preceeded the proofs, resulting in a series of 2,4 sequences. The CMMT analyses of these passages are given in Table 11.

Table 11

CMMT Analyses of Contrived Passages CP7a and CP7b.

List for CP7a.

	Dim. 1	Dim. 2		Weight
Heading	10	1	Words	2
	0	0		2
Explanation	4	3	Illustration, Words, Symbols	10
	4	1		4
	4	2		1
	4	1		6
	4	2		1
	4	1		5
	0	0		1
Generalization	2	1	Words	4
	0	0		4
Explanation	4	3	Illustration, Words	10
	4	1		12
	0	0		2
Generalization	2	1	Words	4
Explanation	4	3	Illustration, Words, Symbols	10
	4	1		7
	4	2		2
	4	1		10
	0	0		3
Generalization	2	1	Words	4
Explanation	4	3	Illustration, Words, Symbols	16
	4	1		8
	4	2		2
	4	1		1
	4	2		1
	4	1		8
	4	2		2
	4	1		2
	4	2		1
	4	1		5
	0	0		2
Generalization	2	1	Words	6
	0	0		2
Heading	10	1	Words	1
	0	0		3
Problems	5	1	Words	2
	0	0		2
	9	1		6
	0	0		2
	9	1		10
	0	0		2
	9	1		6
	0	0		2
	9	1		6
	0	0		2

List for CP7b.

	Dim. 1	Dim. 2		Weight
Heading	10	1	Words	2
	0	0		2
Generalization	2	1	Words	5
	0	0		3
Explanation	4	3	Illustration, Words, Symbols	10
	4	1		4
	4	2		1
	4	1		6
	4	2		1
	4	1		5
	0	0		1
Generalization	2	1	Words	4
Explanation	4	3	Illustration, Words	10
	4	1		11
	0	0		3
Generalization	1	1	Words	4
Explanation	4	3	Illustration, Words, Symbols	10
	4	1		7
	4	2		2
	4	1		10
	0	0		3
Generalization	2	1		5
	0	0		3
Explanation	4	3	Illustration, Words, Symbols	16
	4	1		11
	4	2		2
	4	1		1
	4	2		1
	4	1		13
Heading	10	1	Words	1
	0	0		3
Problems	5	1	Words	2
	0	0		2
	9	1		6
	0	0		2
	9	1		10
	0	0		2
	9	1		6
	0	0		2
	9	1		6
	0	0		2

Table 11. cont.

Dimension 1 interaction matrix for CP7a.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
	1	0	0	0	0	0	0	0	0	0	0
	2	0	14	0	3	0	0	0	0	0	1
	3	0	0	0	0	0	0	0	0	0	0
	4	0	4	0	120	0	0	0	0	0	0
Dimension 1 Categories	5	0	0	0	0	1	0	0	0	1	0
	6	0	0	0	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	27	0
	10	0	0	0	1	1	0	0	0	0	1

Dimension 2 interaction matrix for CP7a.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	107	7	4	0	0	0	0
	2	7	3	0	0	0	0	0
	3	4	0	42	0	0	0	0
Dimension 2 Categories	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Dimension 1 interaction matrix for CP7b.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
	1	0	0	0	0	0	0	0	0	0	0
	2	0	14	0	4	0	0	0	0	0	0
	3	0	0	0	0	0	0	0	0	0	0
	4	0	3	0	117	0	0	0	0	0	1
Dimension 1 Categories	5	0	0	0	0	1	0	0	0	1	0
	6	0	0	0	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	27	0
	10	0	1	0	0	1	0	0	0	0	1

Dimension 2 interaction matrix for CP7b.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	109	5	4	0	0	0	0
	2	5	2	0	0	0	0	0
	3	4	0	42	0	0	0	0
Dimension 2 Categories	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Table 11. cont.

Between Dimension interaction matrix for CP7a.

Dimension 2 Categories

Dimension 1 Categories	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	18	0	0	0	0	0	0
			A				
3	0	0	0	0	0	0	0
4	68	10	46	0	0	0	0
5	2	0	0	0	0	0	0
6	0	0	0	0	0	0	0
			B				
7	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0
9	28	0	0	0	0	0	0
10	3	0	0	0	0	0	0

Between Dimension interaction matrix for CP7b.

Dimension 2 Categories

Dimension 1 Categories	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	18	0	0	0	0	0	0
			A				
3	0	0	0	0	0	0	0
4	68	7	46	0	0	0	0
5	2	0	0	0	0	0	0
6	0	0	0	0	0	0	0
			B				
7	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0
9	28	0	0	0	0	0	0
10	3	0	0	0	0	0	0

CMFT Proportions		Passage CP7a		Passage CP7b	
Proportion of blank space in total passage		.14		.14	
Proportion of each category in non-blank material.		Dim. 1	Dim. 2	Dim. 1	Dim. 2
Category:	1	.00	.68	.00	.69
	2	.10	.06	.10	.04
	3	.00	.26	.00	.27
	4	.71	.00	.71	.00
	5	.01	.00	.01	.00
	6	.00	.00	.00	.00
	7	.00	.00	.00	.00
	8	.00	—	.00	—
	9	.15	—	.16	—
	10	.02	—	.02	—
Proportion of exposition in non-blank material		.82		.81	
Proportion of content development in non-blank material		.82		.81	
Proportion of exercises in content development		.00		.00	
Proportion of exposition in content development		1.00		1.00	
Proportion requiring responses in non-blank material		.17		.17	
Proportion of mathematics illustration in non-blank material		.26		.27	
Proportion of illustration in non-blank material		.26		.27	

The CMMT lists for contrived passages CP7a and CP7b showed the sequential nature of the two passages was as planned. The matrices and the CMMT proportions for both passages were very similar in each case. This demonstrates that the sequential natures of these passages were different but other aspects of the presentation were the same.

Contrived Passages CP9a and CP9b. These passages were contrived from focus passage FP9 which concerned the development of the definitions of cartesian products and relations at the high school level. Passage CP9a was constructed to contain high proportions of definition, generalization, and general explanation and a low proportion of examples. Passage CP9b was constructed to contain low proportions of definition, generalization, and general explanation and a high proportion of examples. In passage CP9a, exercises were integrated with exposition. In passage CP9b, all exercises followed the exposition. Dimension 2 was held constant over both passages. The CMMT analyses for these passages is given in Table 12.

Table 12

CMMT Analyses of Contrived Passages CP9a and CP9b.

List for CP9a.

	Dim. 1	Dim. 2		Weight
Heading	10	1		2
	0	0		2
Examples	3	2		2
	3	1		7
	3	2		2
	3	1		1
	3	2		2
Definition	1	1		4
Examples	3	1		2
	3	2		2
	3	1		1
	3	2		1
	3	1		1
	3	2		2
	3	1		1
	3	2		3
	3	1		1
	3	2		2
	3	1		1
	3	2		1
	0	0		4
Exercises	5	1		4
	7	2		9
	0	0		11
Examples	3	2		4
	0	0		4
	3	2		1
Definition	1	1		3
Heading	3	2		3
	3	1		2
	0	0		7
Exercises	7	1		5
	0	0		3
	7	1		5
	0	0		5
Examples	3	2		4
	0	0		4
	3	2		1
	3	1		3
	3	2		1
	3	1		6
	0	0	Words, Symbols	1
	3	2		1
	3	1		1
	3	2		1
	3	1		2
	0	0		3
	3	2		1
	3	1		8
Definition	1	1		2
	0	0		5
Exercises	7	1		6
	0	0		2
	5	1		6
	0	0		2
	7	2		9
	0	0		15
Examples	3	1		2
	0	0		2
	3	2		3
	0	0		1
	3	2		1
	3	1		5
Definition	1	1		3
Examples	3	1		2
	3	2		2
	3	1		1
	3	2		1
	3	1		2
	0	0		7
Developmental Exercises	6	1		8
	6	2		2
	6	1		2
Exercises	7	1		3
	7	2		1
	7	1		3
	0	0		5

List for CP9b.

	Dim. 1	Dim. 2		Weight
Heading	10	1		2
	0	0		2
Explanation	4	1		15
	0	0		1
Definition	1	1		2
	1	2		2
	1	1		4
	0	0		8
Explanation	4	1		4
Generalization	2	2		3
	0	0		1
Explanation	4	2		5
	0	0		3
	4	2		4
Definition	1	1		2
	1	2		1
	1	1		2
	0	0		7
Explanation	4	1		7
	4	2		1
	4	1		2
Definition	1	1		2
Explanation	4	1		10
	0	0		2
	4	1		1
	4	2		1
	4	1		5
	4	2		1
	4	1	Words, Symbols	4
	4	2		1
	4	1		1
	4	2		1
	4	1		2
	0	0		3
Heading	10	1		1
	0	0		3
Exercises	5	1		4
	7	2		9
	0	0		7
	7	1		5
	0	0		3
	7	1		5
	0	0		3
	7	1		3
	0	0		1
	7	1		3
	0	0		1
	5	1		6
	0	0		2
	7	2		9
	0	0		11
Developmental Exercises	6	1		9
	6	2		2
	6	1		5
Exercises	7	1		3
	7	2		1
	7	1		3
	0	0		1

Table 12. cont.

Dimension 1 interaction matrix for CP9a.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
Dimension 1 Categories	1	8	0	3	0	0	0	1	0	0	0
	2	0	0	0	0	0	0	0	0	0	0
	3	4	0	83	0	1	1	1	0	0	0
	4	0	0	0	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	2	0	0	0
	6	0	0	0	0	0	11	1	0	0	0
	7	0	0	3	0	1	0	36	0	0	0
	8	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	1	0	0	0	0	0	0	1

Dimension 1 interaction matrix for CP9b.

		Dimension 1 Categories									
		1	2	3	4	5	6	7	8	9	10
Dimension 1 Categories	1	12	0	0	3	0	0	0	0	0	0
	2	0	2	0	1	0	0	0	0	0	0
	3	0	0	0	0	0	0	0	0	0	0
	4	3	1	0	60	0	0	0	0	0	1
	5	0	0	0	0	0	0	2	0	0	0
	6	0	0	0	0	0	16	1	0	0	0
	7	0	0	0	0	1	1	30	0	0	0
	8	0	0	0	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0	0	0	0
	10	0	0	0	1	1	0	0	0	0	1

Dimension 2 interaction matrix for CP9a.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
Dimension 2 Categories	1	82	22	0	0	0	0	0
	2	22	40	0	0	0	0	0
	3	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Dimension 2 interaction matrix for CP9b.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
Dimension 2 Categories	1	99	12	0	0	0	0	0
	2	12	29	0	0	0	0	0
	3	0	0	0	0	0	0	0
	4	0	0	0	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
	7	0	0	0	0	0	0	0

Table 12. cont.

Between Dimension interaction matrix for CP9a.

Dimension 2 Categories

		1	2	3	4	5	6	7
	1	12	0	0	0	0	0	0
	2	0	0	0	0	0	0	0
					A			
	3	49	41	0	0	0	0	0
	4	0	0	0	0	0	0	0
Dimension 1 Categories	5	10	0	0	0	0	0	0
	6	10	2	0	0	0	0	0
					B			
	7	22	19	0	0	0	0	0
	8	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	2	0	0	0	0	0	0

Between Dimension interaction matrix for CP9b.

Dimension 2 Categories

		1	2	3	4	5	6	7
	1	12	3	0	0	0	0	0
	2	0	3	0	0	0	0	0
					A			
	3	0	0	0	0	0	0	0
	4	51	14	0	0	0	0	0
Dimension 1 Categories	5	10	0	0	0	0	0	0
	6	14	2	0	0	0	0	0
					B			
	7	22	19	0	0	0	0	0
	8	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	3	0	0	0	0	0	0

CMMT Proportions		Passage CP9a		Passage CP9b	
Proportion of blank space in total passage		.33		.28	
Proportion of each category in non-blank material.		Dim. 1	Dim. 2	Dim. 1	Dim. 2
Category:	1	.07	.63	.10	.73
	2	.00	.37	.02	.27
	3	.54	.00	.00	.00
	4	.00	.00	.43	.00
	5	.06	.00	.07	.00
	6	.07	.00	.11	.00
	7	.24	.00	.26	.00
	8	.00	—	.00	—
	9	.00	—	.00	—
	10	.01	—	.02	—
Proportion of exposition in non-blank material		.61		.55	
Proportion of content development in non-blank material		.69		.65	
Proportion of exercises in content development		.10		.16	
Proportion of exposition in content development		.90		.84	
Proportion requiring responses in non-blank material		.37		.43	
Proportion of mathematics illustration in non-blank material		.00		.00	
Proportion of illustration in non-blank material		.00		.00	

The CMMT lists for contrived passages CP9a and CP9b revealed that exercises were interspersed throughout passage CP9a while they all came at the end of passage CP9b. The lists showed that passage CP9a was high in example content while passage CP9b was high in other types of exposition. The CMMT lists for both passages showed a mixture of word and symbol representation with no illustration.

The dimension 1 interaction matrices showed that passage CP9a contained seven interactions between exposition and exercises while passage CP9b contained only one such interaction. This reflected the plan to integrate exercises with exposition in passage CP9b. The dimension 2 interaction matrices showed the passages were somewhat similar on dimension 2. Only regions A in these two matrices contained entries. Passage CP9a appeared to have more interactions between words and symbols than passage CP9b. This resulted from the integration of the exercises with the exposition in passage CP9a. Regions A of the between dimension matrices also revealed the planned differences in types of exposition between the two passages.

The CMMT proportions showed that passage CP9a contained about 61 percent exposition while passage CP9b contained about 55 percent exposition. Passage CP9a was over 50 percent examples while the exposition in passage CP9b contained no examples. Passage CP9b had 43 percent of its messages requiring responses while passage CP9a had 37 percent of its messages requiring responses. This slight difference resulted from the fact that there was more space devoted to exposition in passage CP9b and not to an absolute difference in the quantities of exercises.

Contrived Passages CP10a and CP10b. These passages were based on focus passage FP10 which was a development of the concept of the mean proportional and the Pythagorean Theorem. Contrived passage CP10a was constructed to contain a high proportion of exposition and a low proportion of exercises and problems. In this passage illustrations were grouped separate from the written text. Contrived passage CP10b was constructed to contain a low proportion of exposition and a high proportion of exercises and problems. Illustrations were integrated with written text. Other aspects were held constant between the two passages. The CMMT analyses of these passages is given in Table 13.

Table 13

CMMT Analyses of Contrived Passages CP10a and CP10b.

List for CP10a.

	Dim. 1	Dim. 2		Weight
Heading	10	1	Words, Symbols	3
	0	0		1
Explanation	4	2		1
	4	1		5
	4	2		1
	4	1		1
	4	2		3
	0	0		1
Definition	1	1		4
	1	2		1
Explanation	4	1		5
	4	2		1
	4	1		2
	0	0		3
Generalization	2	1		9
	0	0		3
Explanation	4	1		1
	4	2		1
	4	1		2
	4	2		1
	4	1		2
	4	2		1
	4	1		6
	4	2		2
	0	0		4
	4	1		1
	4	2		2
	4	1		3
	0	0		2
Generalization	2	1		7
	0	0		1
Explanation	4	1		23
	0	0		1
	4	1		17
	0	0		3
	4	1		6
	4	2		5
	4	1		1
	4	2		4
	4	1		8
Examples	3	1		2
	3	2		1
	3	1		1
	3	2		4
	0	0		4
	3	1		2
	3	2		1
	3	1		1
	3	2		3
	0	0		5
	3	1		5
	3	2		2
	3	1		1
	3	2		3
	0	0		9
Explanation	4	3	Illustration	24
	4	3		60
Heading	10	1	Words, Symbols	1
	0	0		3
Exercises	7	1		5
	7	2		5
	0	0		18
	7	1		8
	0	0		4
	7	3	Illustration	36

List for CP10b.

	Dim. 1	Dim. 2		Weight
Heading	10	1	Words, Symbols	3
	0	0		1
Definition	1	1		2
	1	2		1
	1	1		2
	1	2		1
	1	1		1
	1	2		1
	0	0		4
Generalization	2	1		9
	0	0		4
Explanation	4	3	Illustration	11
	4	1	Words, Symbols	1
	4	2		1
	4	1		2
	4	2		1
	4	1		2
	0	0		5
Generalization	2	1		7
	0	0		2
Explanation	4	3	Illustration	11
	4	1	Words, Symbols	1
	4	2		3
	4	1		1
	4	2		1
	4	1		1
	4	2		5
	4	1		5
	0	0		3
Heading	10	1	Words	1
	0	0		3
Exercises	7	1		5
	7	2	Symbols	1
	0	0		6
	7	2		4
	0	0		12
Application	8	2		1
	0	0		4
	8	3	Illustration	10
	0	0		5
	8	2	Symbols	1
	0	0		2
	8	3	Illustration	8
	0	0		5
	8	2	Symbols	1
	8	3	Illustration	7
	0	0		8
	8	1	Words	5
	0	0		3
	8	3	Illustration	4
	0	0		8
Analysis	9	1	Words	3
	0	0		3
	9	3	Illustration	4
	9	1	Words	5
	0	0		5
	9	1		4
	0	0		4
	9	3	Illustration	4
	0	0		8
	9	1	Words	4
	0	0		4
	9	3	Illustration	4
	0	0		8
	9	1	Words	7
	9	2	Symbols	1
	9	3	Illustration	8
	9	1	Words	9
	3	1		8
	0	0		1
	9	1		3
	0	0		1

Table 13. cont.

Dimension 1 interaction matrix for CP10a.

Dimension 1 Categories

Dimension 1 Categories	1	2	3	4	5	6	7	8	9	10
1	4	0	0	1	0	0	0	0	0	0
2	0	14	0	2	0	0	0	0	0	0
3	0	0	25	1	0	0	0	0	0	0
4	1	2	1	184	0	0	0	0	0	1
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	53	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	1	0	0	1	0	0	2

Dimension 2 interaction matrix for CP10a.

Dimension 2 Categories

Dimension 2 Categories	1	2	3	4	5	6	7
1	112	19	1	0	0	0	0
2	18	23	1	0	0	0	0
3	1	0	118	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0

Dimension 1 interaction matrix for CP10b.

Dimension 1 Categories

Dimension 1 Categories	1	2	3	4	5	6	7	8	9	10
1	7	1	0	0	0	0	0	0	0	0
2	0	14	0	2	0	0	0	0	0	0
3	0	0	7	0	0	0	0	0	1	0
4	0	1	0	44	0	0	0	0	0	1
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	9	1	0	0
8	0	0	0	0	0	0	0	36	1	0
9	0	0	1	0	0	0	0	0	54	0
10	1	0	0	0	0	0	1	0	0	2

Dimension 2 interaction matrix for CP10b.

Dimension 2 Categories

Dimension 2 Categories	1	2	3	4	5	6	7
1	74	10	6	0	0	0	0
2	8	11	4	0	0	0	0
3	8	2	61	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0

Table 13. cont.

Between Dimension interaction matrix for CP10a.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	4	1	0	0	0	0	0
	2	16	0	0	0	0	0	0
					A			
	3	12	14	0	0	0	0	0
Dimension 1 Categories	4	83	22	84	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
					B			
	7	13	5	36	0	0	0	0
	8	0	0	0	0	0	0	0
	9	0	0	0	0	0	0	0
	10	4	0	0	0	0	0	0

Between Dimension interaction matrix for CP10b.

		Dimension 2 Categories						
		1	2	3	4	5	6	7
	1	5	3	0	0	0	0	0
	2	16	0	0	0	0	0	0
					A			
	3	8	0	0	0	0	0	0
Dimension I Categories	4	13	11	22	0	0	0	0
	5	0	0	0	0	0	0	0
	6	0	0	0	0	0	0	0
					B			
	7	5	5	0	0	0	0	0
	8	5	3	29	0	0	0	0
	9	35	1	20	0	0	0	0
	10	4	0	0	0	0	0	0

CMMT Proportions		Passage CP10a		Passage CP10b	
Proportion of blank space in total passage		.17		.37	
Proportion of each category in non-blank material.		Dim. 1	Dim. 2	Dim. 1	Dim. 2
Category:	1	.02	.45	.04	.49
	2	.05	.14	.09	.12
	3	.09	.41	.04	.39
	4	.65	.00	.25	.00
	5	.00	.00	.00	.00
	6	.00	.00	.00	.00
	7	.18	.00	.05	.00
	8	.00	—	.20	—
	9	.00	—	.30	—
	10	.01	—	.02	—
Proportion of exposition in non-blank material		.81		.42	
Proportion of content development in non-blank material		.81		.42	
Proportion of exercises in content development		.00		.00	
Proportion of exposition in content development		1.00		1.00	
Proportion requiring responses in non-blank material		.18		.55	
Proportion of mathematics illustration in non-blank material		.41		.39	
Proportion of illustration in non-blank material		.41		.39	

The CMMT lists for contrived passages CP10a and CP10b demonstrated that passage CP10a contained more exposition than CP10b and that CP10a contained fewer exercises and problems than CP10b. The lists also revealed that illustrations were segregated from the written material in passage CP10a while they were interspersed throughout the written material in passage CP10b.

The dimension 1 interaction matrices also showed the planned differences in quantities of exposition and exercises between the two passages. Region A of the passage CP10a matrix contained 233 entries while region A of the passage CP10b matrix contained 76 entries. Region B of the passage CP10a matrix contained 53 entries all for category 7 while region B of the passage CP10b matrix contained 111 entries of types 7, 8 and 9. The difference between the passages in the placement of the illustrations was shown by the dimension 2 matrices. Passage CP10a contained three interactions between written material and illustration while passage CP10b contained 20 such interactions. The between dimension matrices also showed that passage CP10a contained much more explanation than passage CP10b while passage CP10b contained more and various types of exercises and problems.

The CMMT proportions showed that passage CP10a consisted of 80 percent exposition while passage CP10b consisted of 42 percent exposition. Passage CP10a contained about 18 percent messages requiring responses while passage CP10b contained about 55 percent of such messages. Each passage contained about 40 percent illustration.

Statistical Results

The statistical results of the validity study concern the problem of the number of passages to sample from a section of a textbook, correlational comparisons of passages within and between textbooks, and the identification of presentation variables on which textbooks differ. These results are discussed in this section.

The Sampling Problem. In order to determine if CMMT analyses of entire sections of textbooks can be closely approximated by sampling a small number of CMMT sample passages, successive correlations were made between each entire experimental section and increasingly larger random samples of passages from the section. For these correlations 25 items of data readily available from the CMMT analyses of the passages were used. The items of data and sampling procedure used were detailed in Chapter 4.

Purdue Statistical Program BMD3D (Dixon, 1970) was used to compute the correlations for the sampling problem. These correlations are given in Table 14.

Table 14

Correlations between Increasing Samples and Entire Experimental Sections.

Experimental Section	Number of Passages in Sample								
	1	2	3	4	5	6	7	8	9 or more
ES1	.86	.64	.98	.98	.98	.97	.97	.98	>.98
ES2	.81	.89	.91	.96	.97	.98	.99	.99	1.00
ES3	.97	.95	.97	.97	.99	.99	.99	.99	>.99
ES4	.81	.93	.94	.98	.98	.99	.99	.99	>.99
ES5	.94	.96	.98	.96	.99	.99	.99	.99	1.00
ES6	.99	.98	.99	.99	.99	1.00			
ES7	.94	.99	.98	.99	.99	1.00			
ES8	.92	.96	.98	.99	.99	.99	.99	.99	>.99
ES9	.89	.91	.99	1.00					
ES10	.93	.96	.92	.98	1.00				
ES11	.98	.98	.98	.99	.99	.99	.99	.99	>.99
ES12	.94	.98	.99	.99	.99	.99	1.00		

The correlations given in Table 14 revealed that for all experimental sections high correlations were obtained by sampling only one, two, or three passages. The number of passages it was necessary to sample in order to reach and maintain correlations of .90 and .95 between the samples and their sections is given in Table 15.

Table 15

Number of Passages Necessary to Sample to Reach and Maintain Correlations of .90 and .95 with Entire Sections.

Experimental Section	Total Number of Passages in Section	Sample Size Required for Correlations > .90	Sample Size Required for Correlations > .95
ES1	13	3	3
ES2	9	3	4
ES3	11	1	1
ES4	11	2	4
ES5	9	1	2
ES6	6	1	1
ES7	6	1	2
ES8	11	1	2
ES9	4	2	3
ES10	5	1	2
ES11	10	1	1
ES12	7	1	2

The results given in Table 15 showed that fairly high stable correlations were obtained between the CMMT analyses of entire sections and a sample of only one or two passages. In order to determine more precisely what these correlations mean, further correlational analysis was carried out. The results of this analysis are discussed next.

Correlational Comparisons of Passages. Correlational analysis was used to determine if the CMMT analyses of passages sampled within the same textbook were more closely related than passages sampled from different textbooks. An average within-textbook correlation for each textbook studied was determined by computing the average correlation of all pairs of passages from each experimental section using the same 25 items of CMMT data used in the sampling problem. The average between-textbook correlation was computed by randomly selecting one passage from each of the twelve experimental sections and computing the average correlation between all possible pairs of these passages. The correlations were computed using Purdue Statistical Program BMD3D (Dixon, 1970). Each average correlation was determined by taking the Z transformations of the correlations, computing the mean of these transformed scores, and then taking the inverse Z transformation of the mean (Hays, 1963, p. 608). These average correlations are given in Table 16.

Table 16

Average Correlation of Passages within and between Textbooks.

Average within-Book Correlations

Experimental Section	Average Correlations ($\bar{r}_i$)
ES1	.66
ES2	.61
ES3	.64
ES4	.71
ES5	.88
ES6	.95
ES7	.90
ES8	.85
ES9	.78
ES10	.69
ES11	.93
ES12	.75
Average between-Book Correlation	.53 ($\bar{r}_b$)

The results given in Table 16 indicated that for passages sampled within textbooks, average correlations were consistently higher than for passages sampled from different textbooks. To investigate the significance of this apparent difference a number of statistical tests were performed. First, a λ^2 test suggested by Ostle (1963, p. 227) was used to test for differences among the twelve average within-textbook correlations. The result of this test is given in the first line of Table 17. Next, the test was also used to determine if each average within-textbook correlation differed significantly from the average between-textbook correlation. The results of these tests are also given in Table 17.

Table 17

Results of λ^2 Tests for Comparing Average Correlations between Textbooks.

Hypotheses	df	Observed λ^2	Probability Level
$H_0: \overline{r_1} = \cdots = \overline{r_{12}}$	11	83.89	$p < .01$
$H_0: \overline{r_1} = \overline{r_b}$	1	2.07	$p < .25$
$H_0: \overline{r_2} = \overline{r_b}$	1	.65	$p > .25$
$H_0: \overline{r_3} = \overline{r_b}$	1	1.56	$p < .25$
$H_0: \overline{r_4} = \overline{r_b}$	1	4.14	$p < .05$
$H_0: \overline{r_5} = \overline{r_b}$	1	28.79	$p < .001$
$H_0: \overline{r_6} = \overline{r_b}$	1	70.72	$p < .001$
$H_0: \overline{r_7} = \overline{r_b}$	1	35.62	$p < .001$
$H_0: \overline{r_8} = \overline{r_b}$	1	20.76	$p < .001$
$H_0: \overline{r_9} = \overline{r_b}$	1	9.86	$p < .01$
$H_0: \overline{r_{10}} = \overline{r_b}$	1	3.32	$p < .25$
$H_0: \overline{r_{11}} = \overline{r_b}$	1	52.66	$p < .001$
$H_0: \overline{r_{12}} = \overline{r_b}$	1	7.17	$p < .01$

$\overline{r_i}$ = average within-book correlation for book i, i = 1, 12.

$\overline{r_b}$ = average between-book correlation.

$\lambda^2_{.001}(11) = 31.264$

$\lambda^2_{.25}(1) = 1.32330$ $\lambda^2_{.05}(1) = 3.84146$

$\lambda^2_{.01}(1) = 6.63490$ $\lambda^2_{.001}(1) = 10.828$

Table 17 indicates that the average within-textbook correlations differed at the .01 probability level. This suggests that some textbooks investigated contained more highly similar passages as measured by CMMT analysis than other textbooks. For seven of the textbooks investigated the average within-textbook correlation was significantly different from the between-textbook correlation at the .01 probability level. For the remaining experimental sections the within-textbook correlation did not differ significantly from the between-textbook correlation at the .01 probability level.

Identifying Presentation Variables. In order to determine if the data obtained from CMMT analysis could be used to identify significant differences among the experimental sections, analysis of variance was used. The sixteen variables investigated in this manner were variables which could be readily obtained from the CMMT analyses of the passages in the sections. These variables and the analysis of variance design were detailed in Chapter 4.

For each of the sixteen presentation variables investigated, three statistical tests were performed. First, Bartlett's λ^2 test was used to test for homogeneity of variance (Winer, 1962, p. 95). The test was performed using Purdue Statistical Program DATASUM (Dixon, 1970). The results for each variable are summarized in Table 18.

Table 18

Homogeneity of Variance Tests for Presentation Variables.

Variable	Observed λ^2	Probability Level
1	67.5753	$p < .001$
2	12.5753	$p > .25$
3	20.8815	$p < .05$
4	15.5859	$p < .25$
5	35.5128	$p < .001$
6	37.6567	$p < .001$
7	135.6895	$p < .001$
8	11.1110	$p < .25$
9	21.1336	$p > .05$
10	18.0601	$p > .1$
11	21.4017	$p > .05$
12	80.5406	$p > .001$
13	112.8360	$p > .001$
14	19.5321	$p > .1$
15	81.2449	$p > .001$
16	56.8875	$p > .001$

df = 11 for each variable.

$\lambda^2_{.25}(11) = 13.7007$ $\lambda^2_{.1}(11) = 17.2750$ $\lambda^2_{.05}(11) = 19.6751$

$\lambda^2_{.025}(11) = 21.9200$ $\lambda^2_{.01}(11) = 24.7250$ $\lambda^2_{.001}(11) = 31.264$

The results summarized in Table 18 indicate that on a number of the presentation variables, homogeneity of variance can not be assumed. If the arbitrary significance level of .05 is used then homogeneity of variance can be assumed for only five of the sixteen variables. If the .01 level is used then homogeneity of variance can be assumed for eight of the variables. These results merely show that on a number of the variables investigated there was greater variation among the passages within some books than in others.

The second test performed on each presentation variable was analysis of variance. Even though homogeneity of variance was not found for all of the variables, Winer (1962, p. 920) and Hays (1963, p. 381) indicate that analysis of variance can be performed under these conditions. To perform the analyses, Purdue EDSTAT program ANOVAR (Veldman, 1967) was used. The results for each variable are summarized in Table 19.

The analyses of variance summarized in Table 19 showed that for all of the variables investigated there were significant differences among the means for the twelve books. These results showed that CMMT analysis statistically differentiated among the textbooks on each of the sixteen variables.

In order to determine the extent of the differences among the textbooks, a third test was performed on each variable. The Newman-Keuls Sequential Range Test (Winer, 1962, p. 80) was used to obtain an ordering of the textbooks on each variable and to find how many pairs of means differed

Table 19

Analysis of Variance Tests for Presentation Variables

Variable	$MS_{between}$	MS_{within}	F-ratio	Probability
1	602375.3920	13315.3501	45.24	<.0001
2	260.9897	31.8325	8.20	<.0001
3	47.8005	4.3781	10.92	<.0001
4	19.3675	1.4032	13.80	<.0001
5	24.0855	6.9682	3.46	<.0001
6	157.6088	15.7752	9.99	<.0001
7	54.4608	5.9115	9.21	<.0001
8	73.7919	7.4488	9.77	<.0001
9	428.9311	34.1462	12.56	<.0001
10	2799.2132	238.8240	11.72	<.0001
11	3105.0151	286.4841	10.84	<.0001
12	2408.5381	219.7256	10.96	<.0001
13	3234.0415	409.7675	7.89	<.0001
14	2289.6925	236.2882	9.69	<.0001
15	1099.0162	126.0162	8.72	<.0001
16	1216.9134	171.9584	7.08	<.0001

$df_{between}$ = 11 for each variable.

df_{within} = 90 for each variable.

statistically. For these tests Purdue Statistical Program NKTEST (Dixon, 1970) was used. The results of these tests are summarized in Table 20.

Table 20

Summary of Newman-Keuls Tests for Presentation Variables.

Variable	Number of Pairs of Means Differing at .05 Level.	Number of Pairs of Means Differing at .01 Level.
1	44	40
2	23	20
3	28	26
4	24	20
5	4	2
6	28	20
7	14	11
8	28	18
9	29	24
10	31	25
11	22	18
12	19	18
13	11	11
14	25	20
15	13	11
16	17	10

The Newman-Keuls tests summarized in Table 20 show that on most variables differences were found between many pairs of the twelve books. These results indicate that CMMT analysis could be a powerful tool for differentiating among a number of textbooks on a number of variables.

Results of the Reliability Study

Both between- and within-rater reliability of subjects trained in the use of the CMMT technique were investigated in this study. Three groups of subjects were trained individually and on two occasions coded a set of six CMMT criterion passages. These data were analyzed to determine between- and within-rater differences among the groups and rating difficulties of the criterion passages. The five highest scoring subjects in each group were defined as experts for which between- and within-rater reliability coefficients were determined. The between- and within-rater results are presented below.

Between-Rater Results

Criterion ratings for each message of each passage were determined by computing the most frequent coding of each message by all subjects. The computer program which was used for determining these scores is included in Appendix H. Between-rater scores for each subject on each passage were then computed using the program given in Appendix G. These scores were based on the proportion of agreement between a subject and the criterion ratings of the messages in each passage. Scores for each CMMT dimension were determined in this way. These between-rater scores were used in all analyses of variance presented in this section.

Since the criterion passage scores were used as test scores for analysis of variance it was appropriate to investigate the internal consistency reliability for these scores. To obtain these estimates, Kuder-Richardson Formula 20 coefficients (Nunnally, 1967, pp. 196-197) were computed for each passage on each CMMT dimension. The computer program given in Appendix G was used to compute these coefficients in addition to scoring the passages. These reliability estimates of the between-rater scores are given in Table 21.

Table 21

Internal Consistency Estimates for between-Rater Scores.

Passage	Dimension 1 Estimate	Dimension 2 Estimate
FP1	.91	.77
CP2b	.83	.73
FP8	.84	.82
CP6a	.79	.81
FP12	.71	.65
CP10b	.88	.89

Before performing repeated measures analysis of variance on the two sets of between-rater scores, two checks of homogeneity of variance suggested by Winer (1962, pp. 304-305) were made. First, homogeneity of variance of subjects within the three groups was checked. Second, homogeneity of variance of treatment by subjects within groups was checked. The results of these tests are given in Table 22.

Table 22

Homogeneity of Variance Tests for between-Rater Scores.

Homogeneity of variance of subjects within groups.		
	Dimension 1	Dimension 2
Observed F_{max}	1.58217	6.90149
Probability	> .05	< .01
Homogeneity of variance of treatment by subjects within groups.		
	Dimension 1	Dimension 2
Observed F_{max}	1.37293	1.22812
Probability	> .05	> .05

$F_{max(.01)}$ (3,14) = 5.3

$F_{max(.05)}$ (3,14) = 3.71

$F_{max(.01)}$ (3,70) = 2.1

$F_{max(.05)}$ (3,70) = 1.78

The tests summarized in Table 20 showed that except for subjects within groups on dimension 2 homogeneity of variance could be assumed. Winer (1962, p. 305) indicates that minor violations of the assumptions of homogeneity of variance will not have a great effect on analysis of variance in any event. Thus, it was appropriate to perform the planned analyses of variance. Purdue EDSTAT computer program ANOVAR (Veldman, 1967) was used to perform these analyses on the two sets of between-rater scores. The results of the dimension 1 analysis are given in Table 23.

Table 23

Between-Rater Analysis of Variance for Dimension 1

	Mean Square	df	F-Ratio	Probability
Between Subjects	.0634	44		
Groups	.0076	2	.12	.89
Error (G)	.0660	42		
Within Subjects	.0208	225		
Passages	.1850	5	10.60	.0001
Groups by Passages	.0098	10	.56	.84
Error (P)	.0175	210		

	1	2	3
Group Means	.7140	.707	.6957

	FP1	CP2b	FP8	CP6a	FP12	CP10b
Passage Means	.6635	.6889	.8279	.6704	.6605	.7222

The between-rater analysis of variance summarized in Table 23 shows that on dimension 1 there were no significant differences between groups of raters and no significant interactions between groups and passages. There were, however, significant differences between rating difficulties of the passages. In order to determine which passages differed, the Newman-Keuls procedure suggested by Winer (1962, p. 309) was used. The results of this test are summarized in Table 24.

Table 24

Between-Rater Differences among Passages on Dimension 1.

FP12	FP1	CP6a	CP2b	CP10b	FP8	Ranked Passages
	.0030	.0099	.0284	.0617*	.1674**	FP12
		.0069	.0254	.0587	.1644**	FP1
			.0185	.0518	.1575**	CP6a
				.0333	.1390**	CP2b
					.1057**	CP10b
						FP8

** Significant at .01 level.

* Significant at .05 level.

The results of the Newman-Keuls test summarized in Table 24 show that criterion passage FP8 was easier to code on dimension 1 than any other passage at the .01 level of significance. Criterion passage CP10b was easier to rate on dimension 1 than passage FP12 at the .05 level of significance. No other significant differences between the rating difficulties of the passages was shown. The results of the dimension 2 between-rater analysis of variance are summarized in Table 25.

Table 25

Between-Rater Analysis of Variance for Dimension 2.

	Mean Square	df	F-Ratio	Probability
Between Subjects	.0153	44		
Groups	.0003	2	.02	.98
Error (G)	.0161	42		
Within Subjects	.0056	225		
Passages	.0327	5	6.66	<.0001
Groups by Passages	.0058	10	1.19	.30
Error (P)	.0049	210		

	1	2	3
Group Means:	.9175	.9175	.9142

	FP1	CP2b	FP8	CP6a	FP12	CP10b
Passage Means:	.9000	.8822	.9623	.9107	.9206	.9226

The between-rater analysis of variance summarized in Table 25 shows that on dimension 2 there were no significant differences between groups of raters and no significant interactions between groups and passages. There were, however, significant differences in rating difficulty among the passages. The results of the Newman-Keuls test (Winer, 1962, p. 309) to determine which passages differed are given in Table 26.

Table 26

Between-Rater Differences among Passages on Dimension 2.

CP2b	FP1	CP6a	FP12	CP10b	FP8	Ranked Passages
	.0178	.0285	.0384**	.0404**	.0801**	CP2b
		.0107	.0206	.0226	.0623**	FP1
			.0099	.0119	.0516**	CP6a
				.0020	.0417	FP12
					.0397	CP10b
						FP8

** Significant at .01 level.

The results of the Newman-Keuls test summarized in Table 26 show that criterion passage FP8 was easier to rate on dimension 2 than passages CP2b, FP1, and CP6a at the .01 level of significance. The results also show that passages FP12 and CP10b were easier to rate on dimension 2 than passage CP2b at the .01 significance level.

The five subjects in each group who received the highest overall average between-rater scores were defined to be experts. The data from the experts were used to determine between-rater reliability coefficients. These reliability coefficients are based on the average Scott reliability coefficient for all pairs of experts in each group on each passage. The Scott coefficient which was described in Chapter 3 can be interpreted as the proportion of agreement between subjects with an adjustment for chance agreement. The computer program included in Appendix F was used to compute the between-rater reliabilities which are given in Table 27.

Table 27 revealed somewhat higher between-rater reliabilities on dimension 2 than on dimension 1 of the CMMT system. The table also showed that there was some variation between passages on both dimensions. This variability roughly reflects the results of the analyses of variance.

Table 27

Between-Rater Reliability Coefficients.

	Passage	Dimension 1	Dimension 2
Group 1	FP1	.42	.76
	CP2b	.63	.89
	FP8	.84	.98
	CP6a	.55	.86
	FP12	.40	.85
	CP10b	.72	.94
Group 2	FP1	.58	.73
	CP2b	.52	.77
	FP8	.78	.93
	CP6a	.57	.74
	FP12	.55	.96
	CP10b	.45	.91
Group 3	FP1	.55	.93
	CP2b	.63	.74
	FP8	.81	1.00
	CP6a	.56	.74
	FP12	.52	.83
	CP10b	.60	.81

Within-Rater Results

Since only thirteen of the original fifteen subjects in groups 1 and 2 completed the second coding of the criterion passages, within-rater results are based on groups of size thirteen. In order to make group 3 of size thirteen, two of the original subjects in group 3 were selected at random and dropped from the within-rater analysis.

Within-rater scores were determined for each subject by computing the proportion of agreement between each subject's first and second codings of the criterion passages. The computer program included in Appendix G was used to determine within-rater scores. The internal consistency estimates for the within-rater scores, which were also computed by the program in Appendix G, are given in Table 28.

Table 28

Internal Consistency Estimates for Within-Rater Scores.

Passage	Dimension 1 Estimate	Dimension 2 Estimate
FP1	.87	.80
CP2b	.88	.72
FP8	.90	.83
CP6a	.84	.78
FP12	.91	.84
CP10b	.90	.83

The two checks of homogeneity of variance which were made for the between-rater scores were also made for the within-rater scores before the analyses of variance were performed. The results of these tests are given in Table 29.

Table 29

Homogeneity of Variance Tests for within-Rater Scores.

Homogeneity of variance of subjects within groups.		
	Dimension 1	Dimension 2
Observed F_{max}	1.4945	5.1818
Probability	$> .05$	$> .01$
Homogeneity of variance of treatment by subjects within groups.		
	Dimension 1	Dimension 2
Observed F_{max}	1.1946	1.6207
Probability	$> .05$	$> .05$

$F_{max(.01)}$ (3,12) = 6.1

$F_{max(.05)}$ (3,12) = 4.16

$F_{max(.01)}$ (3,60) = 2.2

$F_{max(.05)}$ (3,60) = 1.85

The tests summarized in Table 29 indicated that homogeneity of variance was a safe assumption in this case. Thus, as in the case of the between-rater data, it was appropriate to perform the planned analyses of variance. Again, Purdue EDSTAT computer program ANOVAR (Veldman, 1967) was used to perform these analyses. Results of the dimension 1 analysis are summarized in Table 30.

Table 30

Within-Rater Analysis of Variance for Dimension 1.

	Mean Square	df	F-Ratio	Probability
Between Subjects	.0889	38		
Groups	.0561	2	.62	.55
Error (G)	.0907	36		
Within Subjects	.0290	195		
Passages	.1377	5	5.29	.0003
Groups by Passages	.0271	10	1.04	.41
Error (P)	.0260	180		

	1	2	3
Group Means:	.6971	.6811	.6447

	FP1	CP2b	FP8	CP6a	FP12	CP10b
Passage Means:	.6636	.6550	.7913	.6338	.6315	.6706

The analysis of variance summarized in Table 30 for within-rater scores was quite similar to the dimension 1 between-rater analysis of variance. Again, no significant differences were found between groups of raters and there were no significant interactions between groups and passages. As in the case of the between-rater results there were significant differences between passages. To determine which passages differed significantly in rating difficulty, the Newman-Keuls test suggested by Winer (1962, p. 309) was used. The results of this test are summarized in Table 31.

Table 31

Within-Rater Differences among Passages on Dimension 1.

FP12	CP6a	CP2b	FP1	CP10b	FP8	Ranked Passages
	.0023	.0235	.0321	.0391	.1598**	FP12
		.0212	.0298	.0368	.1575**	CP6a
			.0086	.0156	.1363**	CP2b
				.0070	.1277**	FP1
					.1207**	CP10b
						FP8

** Significant at .01 level.

As with the between-rater scores, the within-rater results given in Table 31 showed that focus passage FP8 was significantly easier to code on dimension 1 than the other passages. The results of the dimension 2 within-rater analysis of variance which also parallel the between-rater results are given in Table 32.

Table 32

Within-Rater Analysis of Variance for Dimension 2.

	Mean Square	df	F-Ratio	Probability
Between Subjects	.0226	38		
Groups	.0015	2	.06	.94
Error (G)	.0237	36		
Within Subjects	.0071	195		
Passages	.0481	5	7.96	< .0001
Groups by Passages	.0057	10	.94	.50
Error (P)	.0060	180		

	1	2	3
Group Means:	.8940	.8903	.8989

	FP1	CP2b	FP8	CP6a	FP12	CP10b
Passage Means:	.8791	.8467	.9415	.8713	.9224	.9054

The within-rater analysis of variance summarized in Table 32 showed that on dimension 2 there were no significant differences between groups and no significant interactions. There were differences in the rating difficulty of the passages. The results of the Newman-Keuls test (Winer, 1962, p. 309) to determine which passages differed are given in Table 33.

Table 33

Within-Rater Differences among Passages on Dimension 2.

CP2b	CP6a	FP1	CP10b	FP12	FP8	Ranked Passages
	.0246	.0324*	.0587**	.0757**	.0948**	CP2b
		.0078	.0341*	.0511**	.0702**	CP6a
			.0263	.0433*	.0624**	FP1
				.0107	.0361	CP10b
					.0191	FP12
						FP8

** Significant at .01 level.

* Significant at .05 level.

The results of the test summarized in Table 33 are similar to the corresponding between-rater results. Again, passage FP8 appeared to be the easiest passage to code. Passages CP10b and FP12 were easier to code on dimension 2 than passages CP2b, CP6a, and FP1.

Within-rater reliability estimates were determined for the experts in each group using the computer program included in Appendix F. These within-rater coefficients are given in Table 34.

As in the case of the between-rater reliability estimates, the within-rater coefficients tended to be higher on dimension 2 than on dimension 1. Also, a comparison of the between-rater coefficients given in Table 27 with the within-rater coefficients in Table 34 reveals that the within-rater coefficients tended to be slightly higher. Of the eighteen dimension 1 coefficients, fifteen of the within-rater estimates were higher than their corresponding between-rater estimates. On dimension 2 the within-rater and between-rater coefficients appeared to be quite similar. The within-rater coefficients in Table 34 reflected the variation among passages shown in the analyses of variance.

Table 34

Within-Rater Reliability Coefficients.

	Passage	Dimension 1	Dimension 2
Group 1	FP1	.62	.87
	CP2b	.79	.79
	FP8	.90	.95
	CP6a	.67	.88
	FP12	.77	.96
	CP10b	.71	.91
Group 2	FP1	.68	.81
	CP2b	.79	.83
	FP8	.83	.90
	CP6a	.69	.82
	FP12	.61	.88
	CP10b	.78	.88
Group 3	FP1	.66	.99
	CP2b	.58	.72
	FP8	.69	.99
	CP6a	.61	.81
	FP12	.78	.89
	CP10b	.61	.84

Investigator Reliability

To complete the study of reliability a check was made of the reliability of the investigator. This was done by computing the proportion of agreement between the investigator and the between-rater criterion ratings on each criterion passage. In addition a Scott reliability coefficient was computed between the investigator and the criterion ratings on each criterion passages. These proportions and coefficients which were computed using the program in Appendix F are given in Table 35.

Table 35

Investigator Reliabilities.

Passage	Proportion of Agreement		Reliability Coefficients	
	Dimension 1	Dimension 2	Dimension 1	Dimension 2
FP1	.74	1.00	.61	1.00
CP2b	.90	1.00	.88	1.00
FP8	.92	1.00	.88	1.00
CP6a	.70	.96	.61	.93
FP12	.85	1.00	.81	1.00
CP10b	.87	1.00	.83	1.00
Averages	.83	.99	.77	.99

CHAPTER VI

SUMMARY, CONCLUSIONS, AND DISCUSSION

Summary of the Research

In this research a technique for identifying and studying presentation variables in mathematics text was developed and investigated. These presentation variables concern the manner in which authors attempt to communicate mathematics in printed form. The variables were studied in terms of content and representational structures in mathematics text.

The research had purposes which were both developmentally and empirically oriented. From a developmental point of view the purpose of the research was to develop a technique of studying presentation variables in mathematics text. From an empirical point of view the purpose of the research was to investigate the validity and reliability of the technique.

The developed technique has three basic components. The first component consists of a two dimensional system of categories for classifying messages in mathematics text. This category system is called the CMMT category system. Dimension 1 of the CMMT category system which is used to classify messages in terms of mathematical content and process provides subclassifications of the exposition and exercises or problems appearing in mathematics text.

Dimension 2 of the CMMT category system provides a means of classifying the mode of representation used in mathematics text in terms of word, symbol, and illustration classifications.

The second component of the developed technique has to do with procedures for applying the CMMT category system to actual text passages. The basic unit of measure used is one-fourth of a line of print. To apply the category system, a page of text is partitioned into sections conforming to the format of the page. A category number for each dimension of the CMMT system is recorded for each section of this partition and each section is weighted in terms of the unit of measure. Category numbers are recorded following a set of precise definitions and procedures which have been developed.

The third component of the technique developed in this study concerns a method of analyzing the information collected by applying the CMMT categories to mathematics passages. This method of analysis which is called CMMT analysis consists of a sequential listing of categories, a set of matrices for analyzing interactions among categories, and a number of proportions based on the unit of measure. A computer program is used to determine the CMMT analysis of a passage of mathematics text from a rater's codings.

The union of the three basic components of the technique developed in the study is called the CMMT technique. The development of the CMMT technique was an evolutionary process based on practical experience and theoretical considerations. A number of existing mathematics textbooks was examined and tentative category systems were developed, tested, submitted to critical discussion, and revised. As the CMMT technique was being developed a theoretical rationale for the choice of the categories and procedures evolved.

The empirical research consisted of two related studies. The validity study was undertaken to gather evidence of the validity of the CMMT technique as a means of studying presentation variables in mathematics text. The reliability study was undertaken to determine if raters could use the CMMT technique in a consistent and reliable manner.

In the validity study twelve mathematics textbooks from grade four through twelve were investigated. The passages in one section of each of these books were submitted to CMMT analysis. A focus passage was chosen from each section for in depth study. Certain focus passages were used to provide descriptive evidence of the validity of CMMT analysis for describing passages. Other focus passages were used to construct pairs of contrived passages with common content but contrasting forms of organization or presentation. These contrived passages were used to provide further descriptive evidence of the validity of CMMT analysis.

Validity was also studied in a statistical manner. Correlational analysis was used to determine what constituted an adequate sample of passages from textbook sections.

Correlations of passages within textbooks were compared with correlations between passages taken from different textbooks. For all correlations made in the study a set of 25 items of data readily available from the CMMT analyses of investigated passages was used.

To determine if CMMT analysis has validity for identifying presentation variables on which textbooks differ, an analysis of variance model was used. The twelve textbooks investigated were compared on 16 variables readily available from CMMT analyses.

In the reliability study three groups of subjects were investigated. These groups consisted of mathematics education specialists, secondary mathematics teachers, and student teachers of secondary mathematics. The subjects in each group studied a training booklet to learn the CMMT technique and then rated six criterion passages on two obcasions separated by a period of a minimum of four weeks. Repeated measures analysis of variance was used to determine if within- and between-rater differences existed between groups of subjects and to determine if differences existed in the rating difficulty of the criterion passages.

To complete the reliability study a number of reliability coefficients were computed. These included within- and between-rater coefficients for each CMMT dimension and each group as well as reliability estimates for the investigator.

Conclusions and Discussion

In this section conclusions and discussion concerning the developmental, validity, and reliability aspects of the research are given.

Developmental Aspects

The development of the CMMT technique must be viewed with two limitations in mind. First, the technique is essentially new. Although similar instruments have been used to study classroom interaction, no other researcher has attempted to use such a technique in a setting similar to this study. Thus, the CMMT technique is not directly a part of any existing theory or body of research. Second, the technique is largely the work of a single researcher. While the investigator did discuss the technique with a number of researchers as it was being developed, the technique does not represent the combined efforts of several equally contributing and involved researchers. To overcome these limitations, a number of other researchers will have to become involved in studying and further developing the technique. A solid theoretical rationale on which to base the technique will have to be established.

Before the CMMT technique can be regarded as a practical and usable technique, researchers must deal with several developmental issues. These issues concern the CMMT category system itself, the procedures for applying the system to text materials, and CMMT analysis. Developmental issues related to the category system concern the problems of the frequency with which the CMMT categories appear in actual text materials, the continuous nature of some of the classifications, and the mutual exclusiveness of the categories.

The problem of the frequency with which classifications appear in actual mathematics textbooks is illustrated by dimension 1 of the CMMT system. By far the most numerous categorizations of the text materials investigated consisted of specific and general explanation (categories 3 and 4) and exercises (category 7). The other categories tended to appear relatively infrequently. The possibility of combining some of the rarely occurring categories such as definition (category 1) and generalization (category 2) needs to be investigated. Perhaps some of the most frequently occurring categories need to be further subdivided. This may be particularly true of general explanation (category 4) which seemed to become a catchall for expository material.

The continuous nature of the dichotomy between some of the pairs of categories presents another problem on which more developmental effort is needed. The categories of specific and general explanation (categories 3 and 4 of dimension 1) illustrate this problem. It is difficult to determine at what point something is no longer specific and becomes general. Other categories presenting this problem include the exercise and problem classifications on dimension 1 and the illustration classifications on dimension 2.

Effort is needed in the development of more precise definitions of the categories to overcome this problem.

Mutual exclusiveness also appears as a possible problem in the exercise and problem categories on dimension 1. For example real world problems can be of a rote or practice nature and thus classifiable as both exercises (category 7) and application (category 8). Such problems could also play a role in the development of the material and thus be classified in category 6. Further developmental effort is needed to insure the mutual exclusiveness of the categories.

A developmental issue related to the procedures used in applying the category system to actual text materials concerns the time required to code a page. The investigator found the task of subdividing a page, weighting the partitions, coding the material, and transferring the data to computer cards for analysis to be an extremely time consuming process. Although researchers may be willing to spend such amounts of time, it is doubtful that anyone else would. Thus, the technique as it now stands is limited in this respect. Further developmental effort could be directed toward reducing the time requirement.

CMMT analysis also needs further developmental attention. The meaningfulness of the area unit is questionable. Some types of messages such as problems may pack much more into a unit of measure than other messages such as general explanation. A symbolically written message ordinarily uses less space than the same message written out in words. Illustrations tend to take up large areas. The same illustration could exist in two passages but be of vastly different sizes thus yielding different CMMT proportions. In light of this the sequential and interaction aspects of CMMT analysis appear to be more meaningful than the CMMT proportions. In any event, future research may lead to the development of some other unit of measure.

There is much developmental work still to be done if the CMMT technique is to become widely used. Whether or not this work should be done will depend on researchers' judgements of the potential of the technique. These judgements should be made in terms of the empirical evidence supplied by the validity and reliability studies. The conclusions which can be drawn from these studies are discussed in the following sections.

The Validity Study

The following conclusions may be drawn from the results of the validity study.

1. CMMT analysis can be used to describe the organization of mathematics text passages.
2. CMMT analysis reflects similarities and differences between pairs of passages with known organizational makeups.
3. CMMT analyses of entire sections of mathematics textbooks correlate highly with the CMMT analyses

of random samples of two or three passages from the sections.

4. CMMT analyses of passages sampled within textbooks tend to correlate slightly higher than CMMT analyses of passages sampled from different textbooks.
5. CMMT analyses of textbooks measure variables which statistically differentiate among textbooks.

The above conclusions should be viewed within the bounds of certain limitations. Conclusions 1 and 2 are based only on descriptive information. It is the investigator's opinion that the descriptive results given in Chapter 5 support these conclusions.

It is not entirely clear what the correlational conclusions drawn in 3 means in terms of sampling passages from textbook sections for CMMT analysis. While the correlations of sections with two or three sampled passages tend to be above .90 it is difficult to say if this is high enough. Conclusion 4 indicates that caution should be exercised in using a small sample of passages to represent sections of textbooks. While correlations among passages within textbooks tend to be fairly high in some cases, these correlations do not seem to be a great deal higher than correlations among passages from different textbooks.

Certain types of passages within some of the textbooks investigated tended to correlate lowly with other passages in the same book. For example, review passages or passages consisting only of verbal problems had very low or even negative correlations with the majority of passages within some texts. Perhaps the average within-textbook correlations for the different textbooks serve only to give an indication of the consistency of presentation in the books. Whether this level consistency is important must be determined by further research.

Conclusion 5 may be the most important conclusion drawn from the validity study. Any valid measurement instrument must be able to differentiate among the things it measures, if it is to have any use. CMMT analysis did differentiate among the textbooks in this study on all sixteen variables investigated. It remains for further research to determine if these differences have meaning in terms of any learner variables.

In view of the limitations discussed above it is the opinion of the investigator that the conclusions of the validity study support the contention that the CMMT technique has potential for becoming a valid means of studying mathematics text.

The Reliability Study

The following conclusions may be drawn from the results of the reliability study.

1. In terms of both between- and within-rater scores, groups were statistically equivalent on both dimensions of the CMMT category system.

2. In terms of both between- and within-rater scores, some passages were significantly more difficult to code than other passages on both CMMT dimensions.
3. Most between-rater reliability coefficients on dimension 1 ranged between .50 and .80.
4. Most between-rater reliability coefficients on dimension 2 ranged between .75 and 1.00.
5. Most within-rater reliability coefficients on dimension 1 ranged between .60 and .90.
6. Most within-rater reliability coefficients on dimension 2 ranged between .75 and 1.00.
7. Investigator reliability measured in terms of codings of the subjects averaged about .77 on dimension 1 and .99 on dimension 2.

These conclusions must also be viewed within the limitations of the reliability study. It should be kept in mind that subjects in this study trained themselves in the use of the CMMT technique and were not requested to spend large amounts of time on the project. With improved training techniques it would not be unreasonable to expect raters to achieve higher levels of reliability. The slightly higher within-rater reliabilities achieved on dimension 1 seem to support this.

The lack of differences between groups may also be a reflection of the individualized training and the short training period. Possibly no group of subjects was trained long enough for the differences in experience and education among the groups to have any effect. It is also a possibility that a lack of motivation on the part of all subjects to spend five or six hours on the required task had a leveling effect on the results.

The reliability coefficients of the investigator's codings must be interpreted cautiously. It is the feeling of the investigator that the dimension 1 average estimate of .77 is low. This coefficient was computed in terms of the most frequent ratings made by the subjects in the experiment. Some of these ratings were not based on an overwhelming concensus of the subjects. In fact, many were based on a minority opinion. It is the opinion of the investigator that in most cases where he disagreed with the most frequent ratings he was right and the subjects were wrong.

It remains for further research to determine what levels of reliability can be obtained by subjects with improved and prolonged training. It also remains for research to determine what levels of reliability are necessary for the technique to be of practical value. It is the opinion of the investigator that the conclusions drawn from the results of the reliability study indicate that the CMMT technique has potential for becoming a reliable technique for studying mathematics text.

Recommendations for Further Research

There is a clear need for research which systematically studies mathematics text. Techniques for doing this in a meaningful way are not plentiful. The CMMT technique represents an attempt to develop one technique for studying presentation variables in mathematics text. The potential of the technique as a useful means for studying mathematics text is the crucial factor. Further reserach will have to determine if its potential will be realized. Recommendations for the directions such research should take are given in this section.

Before the CMMT technique can be used to undertake studies of mathematics text, the developmental and reliability problems discussed in the previous section should be resolved. Studies investigating these problems should involve a number of researchers and subjects over a period of time. In these studies developmental problems could be investigated in conjunction with investigations of reliability. Researchers should work directly with subjects to determine where coding difficulties are encountered. Training procedures should be investigated in terms of what levels of reliability can be obtained in given time intervals.

After the developmental and reliability issues have been satisfactorily settled, researchers will be able to undertake actual studies of mathematics text. These studies could take a number of forms. Descriptive studies could be carried out in which the nature of the presentation in various textbooks is merely described in terms of CMMT analysis. Comparative studies of the presentation in various textbooks in terms of CMMT analysis could also be carried out. These descriptive and comparative studies, while providing information about the presentation in particular textbooks, could also seek to operationalize in terms of CMMT analysis the meaning of such terms as "discovery" or "conventional" approaches to presenting mathematics text.

Studies in which connections between learner variables and CMMT analysis patterns are investigated will need to be undertaken. Such studies could attempt to explain some of the results of existing comparative studies of mathematics text, such as the National Longitudinal Study, in terms of CMMT analysis. A regression analysis model could be used in an attempt to identify certain CMMT patterns which produced superior achievement or attitudinal results in existing studies.

Experimental studies could compare learner outcomes on text passages which are constructed to have given CMMT analysis patterns. These studies could search for patterns of presentation which produce specific learner outcomes such as superior performance of basic skills, superior problem solving, positive attitudes toward studying mathematics textbooks, etc. The patterns investigated in these studies will have to be chosen on some theoretical basis and/or in terms of the outcomes of the types of studies described in the previous two paragraphs.

Finally, educators may wish to put a fully developed CMMT technique to practical use. Textbook authors may wish to use CMMT analysis as a tool in the construction of mathematics text materials. Publishers may wish to use CMMT analysis information to help in describing their textbooks to educators. Textbook selection committees may wish to use the CMMT analyses of the textbooks they are considering to aid them in their decisions. Teacher education programs in mathematics may wish to make use of the CMMT technique in training teachers in the evaluation of mathematics text materials. These practical applications should be made only after positive results of further research on the CMMT technique have been received.

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APPENDIX A

RATER TRAINING BOOKLET

Content

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I. Introduction to the CMMT Categories

The CMMT category system is a system of categories for classifying messages which appear in mathematics text. CMMT stands for "Classification of Messages in Mathematics Text." The CMMT category system which is two dimensional is given below.

CMMT Category System

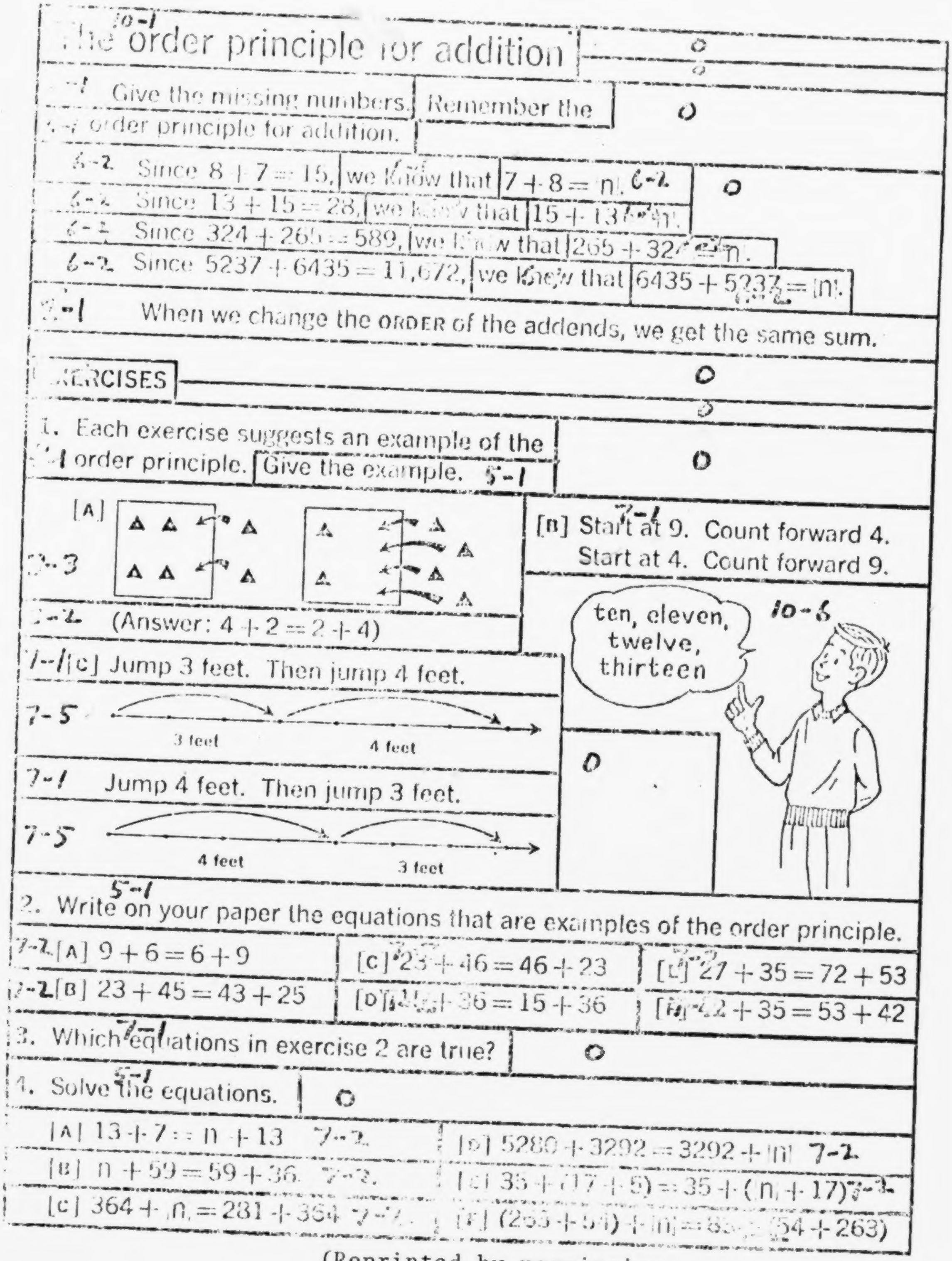
Dimension 1: Content Mode

	0. Blank Space	
Messages requiring only reception.	1. Definition (Meanings of words or symbols.)	Exposition
	2. Generalizations (Important rules, axioms, theorems, formulas, etc.)	
	3. Specific Explanation (Concrete examples and discussion in specific terms.)	
	4. General Explanation (Proofs of propositions, general discussion, etc.)	
Messages calling for responses other than reception.	5. Procedural Instructions (Directions.)	Subclassifications of questions, exercises, problems, activities, etc.
	6. Developing Content (Questions in exposition, developmental activities, guided discovery exercises, etc.)	
	7. Understanding Developed Content (Exercises involving routine computations, practice, identification, etc.)	
	8. Applying Developed Content (Real world problems, applications of generalizations in concrete situations, etc.)	
	9. Analyzing and Synthesizing Developed Content (Proving propositions, finding new relationships in developed content, unguided discovery, etc.)	
	10. Other Material (Headings, non-mathematical materials, etc.)	

Dimension 2: Presentation Mode

	0. Blank Space
Written text.	1. Words 2. Mathematical Symbols
Illustration	3. Representations of Abstract Ideas (Venn diagrams, geometric diagrams, mapping pictures, etc.) 4. Graphs (Number lines, coordinate graphs, bar graphs, etc.) 5. Representations of Physical Objects or Situations (Plans, maps, cross sectional drawings, photographs, etc.) 6. Non-mathematical Illustrations (Motivational photographs, cartoons, etc.)
	7. Combinations of Illustrations with Written Text (Flow charts, mathematical tables, tree diagrams, etc.)

The CMMT category system is used to classify messages in mathematical passages. To do this, the passage is partitioned into units determined by a set of rules and the format of the passage. Each unit is then classified with respect to both dimensions of the CMMT category system. The following page is an example of a page of text which has been rated using the CMMT technique. The symbol m-n recorded in a given unit indicates that the message in that unit is classified as category m on dimension 1 and category n on dimension 2.



10-1 The order principle for addition 0

-1 Give the missing numbers. Remember the order principle for addition. 0

6-2 Since 8 + 7 = 15, we know that 7 + 8 = n. 6-2 0

6-2 Since 13 + 15 = 28, we know that 15 + 13 = n.

6-2 Since 324 + 265 = 589, we know that 265 + 324 = n.

6-2 Since 5237 + 6435 = 11,672, we know that 6435 + 5237 = n. 6-2

7-1 When we change the ORDER of the addends, we get the same sum.

EXERCISES 0

1. Each exercise suggests an example of the order principle. Give the example. 5-1 0

7-3 [A]

7-2 (Answer: 4 + 2 = 2 + 4)

7-1 [B] Start at 9. Count forward 4.
Start at 4. Count forward 9.

10-6

ten, eleven, twelve, thirteen

7-1 [C] Jump 3 feet. Then jump 4 feet.

7-5 3 feet 4 feet

0

7-1 Jump 4 feet. Then jump 3 feet.

7-5 4 feet 3 feet

5-1

2. Write on your paper the equations that are examples of the order principle.

7-2 [A] 9 + 6 = 6 + 9
7-2 [B] 23 + 45 = 43 + 25
7-2 [C] 23 + 46 = 46 + 23
7-2 [D] 15 + 36 = 15 + 36
7-2 [E] 27 + 35 = 72 + 53
[F] 42 + 35 = 53 + 42

7-1

3. Which equations in exercise 2 are true? 0

5-1

4. Solve the equations. 0

[A] 13 + 7 = n + 13 7-2
[B] n + 59 = 59 + 36. 7-2
[C] 364 + n = 281 + 364 7-2
[D] 5280 + 3292 = 3292 + n 7-2
[E] 35 + (17 + 5) = 35 + (n + 17) 7-3
[F] (263 + 54) + n = 85 + (54 + 263)

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Eicholz and O'Daffer, Addison-Wesley
Publishing Company, 1968.)

When the CMMT classification of all messages in a passage is completed the information may be analyzed in ways which reflect both sequential and quantitative aspects of the organization of the passage. Sequential aspects are represented by making an ordered list of the classifications following the natural flow of the printed material. A matrix similar to those used in Flanders' interaction analysis is used to analyze the nature of the interactions among CMMT categories. Quantitative aspects of a passage are described by determining the proportion of messages in the various categories and in logical combinations of categories. All of the descriptive information for a given passage is called the CMMT analysis for that passage. The CMMT analysis for the passage on page 3 follows.

List of Messages in Order of Occurrence

Dim. 1	Dim. 2	Weight
10	1	3
0	0	1
5	1	2
4	1	3
0	0	3
6	2	1
6	1	1
6	2	2
6	1	1
6	2	2
6	1	1
6	2	3
6	1	1
6	2	1
0	0	3
2	1	4
0	0	4
10	1	1
0	0	3
4	1	3
5	1	1
0	0	4
3	3	6
3	2	2
7	1	4
10	6	12
7	1	2
7	5	4
7	1	2
7	5	4
0	0	4
5	1	4
7	2	8
7	1	2
0	0	2
5	1	1
0	0	3
7	2	12

(The weight indicates the number of area units of approximately 1/4 of a line of print devoted to each message recorded in the list.)

End

Frequency Matrix for Dimension 1

0	0	0	0	0	0	0	0	0	0
0	3	0	0	0	0	0	0	0	1
0	0	7	0	0	0	1	0	0	0
0	0	0	4	1	1	0	0	0	0
0	0	1	1	4	0	2	0	0	0
0	1	0	0	0	12	0	0	0	0
0	0	0	0	2	0	34	0	0	1
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	0	1	0	0	13

(The entry n in row i and column j indicates message type i was followed by message type j, n times in the passage. Blank space is ignored in this matrix.)

Proportion in Each Category for Dimension 1

Category	Proportion
0	.225
1	0
2	.043
3	.087
4	.065
5	.087
6	.141
7	.402
8	0
9	0
10	.174

Proportion Exposition .196

Proportion Content Development (1 thru 4 plus 6) .337

Proportion Requiring Responses in Content Development .419

Proportion of Exposition in Content Development .581

Proportion Requiring Responses .630

Frequency Matrix for Dimension 2

25	7	1	0	2	1	0
7	23	0	0	0	0	0
0	1	5	0	0	0	0
0	0	0	0	0	0	0
2	0	0	0	6	0	0
1	0	0	0	0	11	0
0	0	0	0	0	0	0

Proportion in Each Category for Dimension 2

Category	Proportion
0	.225
1	.391
2	.326
3	.065
4	0
5	.087
6	.130
7	0

Proportion Mathematics Illustration .152

Proportion Illustration .283

Interaction Matrix Dimension 1 and 2

0	0	0	0	0	0	0
4	0	0	0	0	0	0
0	2	6	0	0	0	0
6	0	0	0	0	0	0
8	0	0	0	0	0	0
4	9	0	0	0	0	0
10	20	0	0	8	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
4	0	0	0	0	12	0

(The entry n in row i and column j indicates message type i on dimension 1 was of type j on dimension 2, n times in the passage.)

The information about a mathematics passage which is provided by CMMT analysis could be used for a number of purposes involving both practical application and research in mathematics education.

1. CMMT provides a means for describing mathematics text in specific terms. Such information could be useful to textbook authors, publishers, and textbook selection committees.
2. CMMT provides a means of manipulating and controlling textual variables in experimental studies using printed mathematics material. Thus it could be a useful instrument for researchers studying mathematics text materials.

The purpose of subsequent sections of this booklet is to provide the reader training in the use of the CMMT technique for classifying messages in mathematics text. Complete descriptions of all of the categories with examples and general procedural techniques are given in the following sections of the booklet. The final section includes numerous keyed practice passages which the reader may use in training himself in the use of the CMMT technique.

II. Description of Dimension 1 Categories

0. Blank Space: Any unit which is completely blank.
1. Definition (Meanings of words or symbols.): Messages whose primary purpose is to give the mathematical meanings of words, symbols, or phrases. Any statement of a definition or any description of a mathematical term. May be integrated into the written text or set off as a formal definition.
2. Generalization (Important rules, axioms, theorems, formulas, etc.): Messages whose primary purpose is to give important general mathematical concepts, ideas, or procedures. Any statement of an axiom or theorem. May be integrated into the written text or formally set off as a rule, formula, principle, etc.
3. Specific Explanation (Concrete examples and discussion in specific terms.): Messages whose primary purpose is to give specific or concrete instances of general concepts, ideas, or procedures. Specific examples of general definitions, axioms, or theorems. Any discussion or explanation of concrete examples. May be integrated into written text or specifically set off as formal examples.
4. General Explanation (Proofs of propositions, general discussion, etc.): Messages whose primary purpose is to explain or discuss general concepts, ideas or procedures. Any proof of a formal proposition. General

statements which seek to clarify, justify, show how or why, etc. May be used to introduce, relate, or summarize aspects of a passage. Expository statements with mathematical content which are not clearly classifiable as 1 through 3.

5. Procedural Instructions (Directions.): Procedural directions appearing anywhere in a text passage. May indicate to the reader what he is to do with a set of exercises or problems. May be stated in the form of a question. Does not apply to directive statements containing substantial mathematical information.

   Typical examples:

   a. Solve the following equations.
   b. Complete.
   c. Look at Figure 7.
   d. What are the solution sets for the open sentences below?

6. Developing Content (Questions in exposition, developmental activities, guided discovery exercises, etc.): Questions, activities, or exercises designed to help develop content. May call for verbal discussion, physical activity, or thought or written responses. May be designed to guide the student to discover some mathematical concept. Exercises or problems whose primary purpose is to develop or present new content. Questions, activities, or exercises integrated into the

exposition which are designed to make the reader participate in the development of the content.

7. Understanding Developed Content (Exercises involving routine computation, practice, identification, etc.): Exercises which are clearly on the knowledge or comprehension level of Bloom's Taxonomy. May involve lowest level of application where a generalization is used in a situation essentially identical to the situation in which the generalization was presented in the text. The emphasis here is on understanding the content which was presented in the text. May involve the recall of specific information and procedures or the understanding of presented content with only routine alterations of material. May involve the use of material without relating it to other materials or situations or seeing its broader implications.

   Typical examples:

   a. Exercises of a purely computational nature.
   b. Exercises intended to give practice in a skill or procedure.
   c. Exercises which are essentially the same as examples given in the text. That is, exercises whose solutions depend on the imitation of examples worked out in the text.
   d. Problems which require only a direct routine application of a definition or generalization.

Such as finding the circumference of a circle with a given radius. (Where the emphasis is on understanding the generalization rather than on using it to find solutions to problems.)

8. Applying Developed Content (Real world problems, applications of generalizations in concrete situations, etc.): Exercises or problems which are on the application level of Bloom's Taxonomy. The emphasis here is on using the content presented in the text in a new or different situation. The new situation may be mathematical or a real life or physical situation. May involve the use of an abstraction in a concrete situation which the student has not seen before. Involves the recall and understanding of information and in addition the utilization of the content in a non-routine manner to find the answer to a problem.

   Typical examples:

   a. Using general formulas or procedures to solve mathematical problems.
   b. Using general mathematical principles developed in the text to find solutions to real life problems. (Word problems.)
   c. Applying general definitions, axioms, and theorems in concrete situations or to specific mathematical models.
   d. Interpreting concrete examples in terms of general structures.

9. Analyzing and Synthesizing Developed Content (Proving propositions, finding new relationships in developed content, unguided discovery, etc.): Problems which are on the analysis or synthesis level of Bloom's Taxonomy. May involve the breakdown of content into its basic elements so that relationships between ideas are made more explicit.

   Typical examples:

   a. Problems requiring the proof or disproof of a proposition.
   b. Problems requiring the discovery of new relationships between elements of developed content.

10. Other Material (Headings, non-mathematical materials, etc.): Messages not fitting categories 0 through 9. May be headings of sections, exercises, etc. May be motivational or historical material. Messages which are non-mathematical in character.

III. Notes on Using Dimension 1 Categories

A. Begin by mentally delineating between the blank space (0), exposition (1 - 4), messages requiring responses (5 - 9), and other material (10). This will make the task of making finer distinctions easier.

B. To rate exposition (1 - 4):

1. First eliminate definitions (1) and generalizations (2).
2. Of the remaining exposition, classify all messages giving specific instances of more general content as specific explanation (3).
3. Classify the remaining exposition as general explanation (4).

C. To rate messages requiring responses (5 - 9):

1. First eliminate procedural instructions (5).
2. Base decisions involving exercises and problems (6 - 9) on the processes involved. Do not base these decisions on the difficulty of the problem or exercise.

D. Illustrations as well as written messages must be classified on dimension 1. The classification of an illustrative message is determined by the same rules as the classification of written messages. If a diagram illustrates a definition it is classified as a definition, if it illustrates a proof it is classified as general explanation, etc.

E. When in doubt about how to classify a message reread the descriptions of the appropriate categories.

IV. Description of Dimension 2 Categories

0. Blank Space: Any unit which is completely blank.
1. Words: Messages which are made up predominately of ordinary English words.
2. Mathematical Symbols: Messages which are made up predominately of mathematical symbols.
3. Representations of Abstract Ideas: May be drawings to illustrate abstract sets, Venn diagrams, mapping pictures, geometric diagrams, etc.
4. Graphs: May be bar graphs, line graphs, circle graphs, etc. Or, may be number lines or various types of coordinate system graphs.
5. Representations of Physical Objects or Situations: Must have some mathematical content. May be drawings or photographs of real objects or living things which illustrate mathematical content. Could be cross sectional diagrams, maps, plans, charts, etc.
6. Non-mathematical Illustrations: Non-mathematical in nature and not illustrating any mathematical content being considered. May be motivational or historical cartoons, photographs, or drawings.
7. Combinations of Illustrations with Written Text: May be some sort of diagram which contains substantial mathematical information in the form of words or symbols. Includes all mathematical tables, flow charts, tree diagrams, etc.

V. Notes on Using Dimension 2 Categories

A. To decide if a written message in a unit is predominately words or symbols, count the words necessary to say the message. If the symbols account for more than or the same number of these words as the ordinary English then classify the message as symbols (2). Otherwise, classify the message as words (1).

B. When classifying messages (3 - 7) eliminate the non-mathematical illustrations (6) first. Then make the remaining classifications. When in doubt reread the appropriate category descriptions.

VI. Procedural Suggestions for Rating Passages

A. Before beginning to rate a passage get a broad idea of its general nature. Read through the passage once to get an understanding of the content.

B. When learning how to rate materials it may be helpful to make all ratings for dimension 1 first and then go through the passage again for dimension 2 classifications.

C. When making decisions consider the context in which the message appears. Decide what purpose the message is playing in the passage. If a unit appears to fit more than one category of a dimension decide which category is predominate. Do not change from one unit to the next when in doubt whether or not the classification should be changed.

D. Refer back to the appropriate descriptions of categories and notes in sections II through V when in doubt. Use the rules given to make all decisions. Do not invent new rules of your own to follow.

VII. Practice Passages and Keys

Instructions for Using Practice Passages

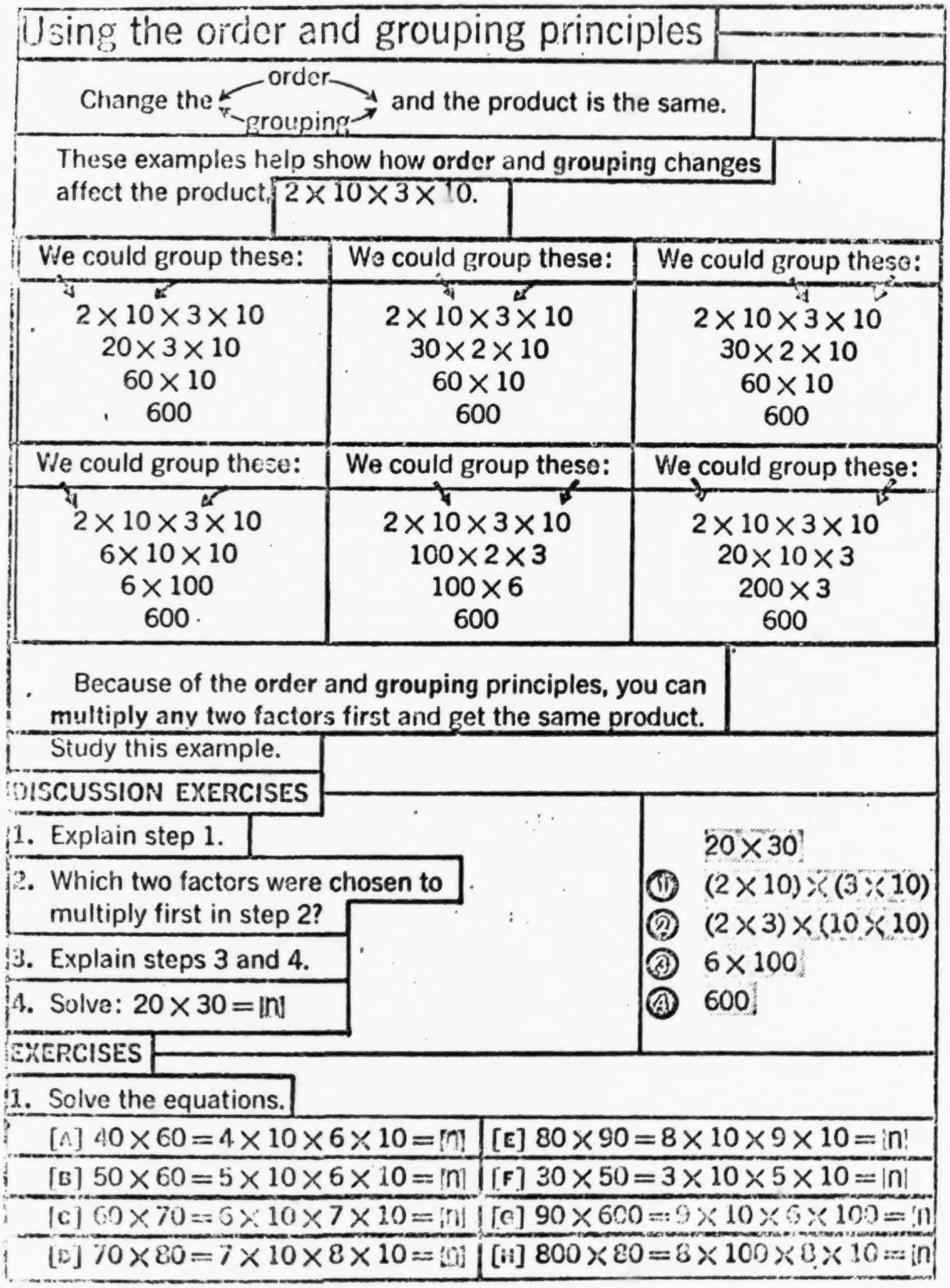
A. Rate each practice passage in this section by classifying each unit with respect to both dimensions of the CMMT category system as in the example passage in section I. Record the appropriate symbol of the form m-n in each unit. Refer back to any materials in this booklet as needed.

B. When you have completed rating a practice passage check your ratings against the keyed passage on the page following the practice passage.

C. Rate practice passages until you feel you have mastered the CMMT technique. If you agree with the key more than 90% of the time you should feel fairly comfortable. Rate practice passages until you have achieved the 90% level.

D. When you have mastered the CMMT technique rate the passages in the CMMT Technique: Experimental Passage Booklet.

Practice Passage 1



Using the order and grouping principles

Change the $\overset{\text{order}}{\underset{\text{grouping}}{\rightleftarrows}}$ and the product is the same.

These examples help show how **order** and **grouping** changes affect the product, 2 × 10 × 3 × 10.

We could group these:	We could group these:	We could group these:
2 × 10 × 3 × 10 20 × 3 × 10 60 × 10 600	2 × 10 × 3 × 10 30 × 2 × 10 60 × 10 600	2 × 10 × 3 × 10 30 × 2 × 10 60 × 10 600
We could group these:	We could group these:	We could group these:
2 × 10 × 3 × 10 6 × 10 × 10 6 × 100 600	2 × 10 × 3 × 10 100 × 2 × 3 100 × 6 600	2 × 10 × 3 × 10 20 × 10 × 3 200 × 3 600

Because of the **order** and **grouping** principles, you can multiply any two factors first and get the same product.

Study this example.

20 × 30
① (2 × 10) × (3 × 10)
② (2 × 3) × (10 × 10)
③ 6 × 100
④ 600

DISCUSSION EXERCISES

1. Explain step 1.
2. Which two factors were chosen to multiply first in step 2?
3. Explain steps 3 and 4.
4. Solve: 20 × 30 = $\boxed{n}$

EXERCISES

1. Solve the equations.

[A] 40 × 60 = 4 × 10 × 6 × 10 = $\boxed{n}$	[E] 80 × 90 = 8 × 10 × 9 × 10 = $\boxed{n}$
[B] 50 × 60 = 5 × 10 × 6 × 10 = $\boxed{n}$	[F] 30 × 50 = 3 × 10 × 5 × 10 = $\boxed{n}$
[C] 60 × 70 = 6 × 10 × 7 × 10 = $\boxed{n}$	[G] 90 × 600 = 9 × 10 × 6 × 100 = $\boxed{n}$
[D] 70 × 80 = 7 × 10 × 8 × 10 = $\boxed{n}$	[H] 800 × 80 = 8 × 100 × 8 × 10 = $\boxed{n}$

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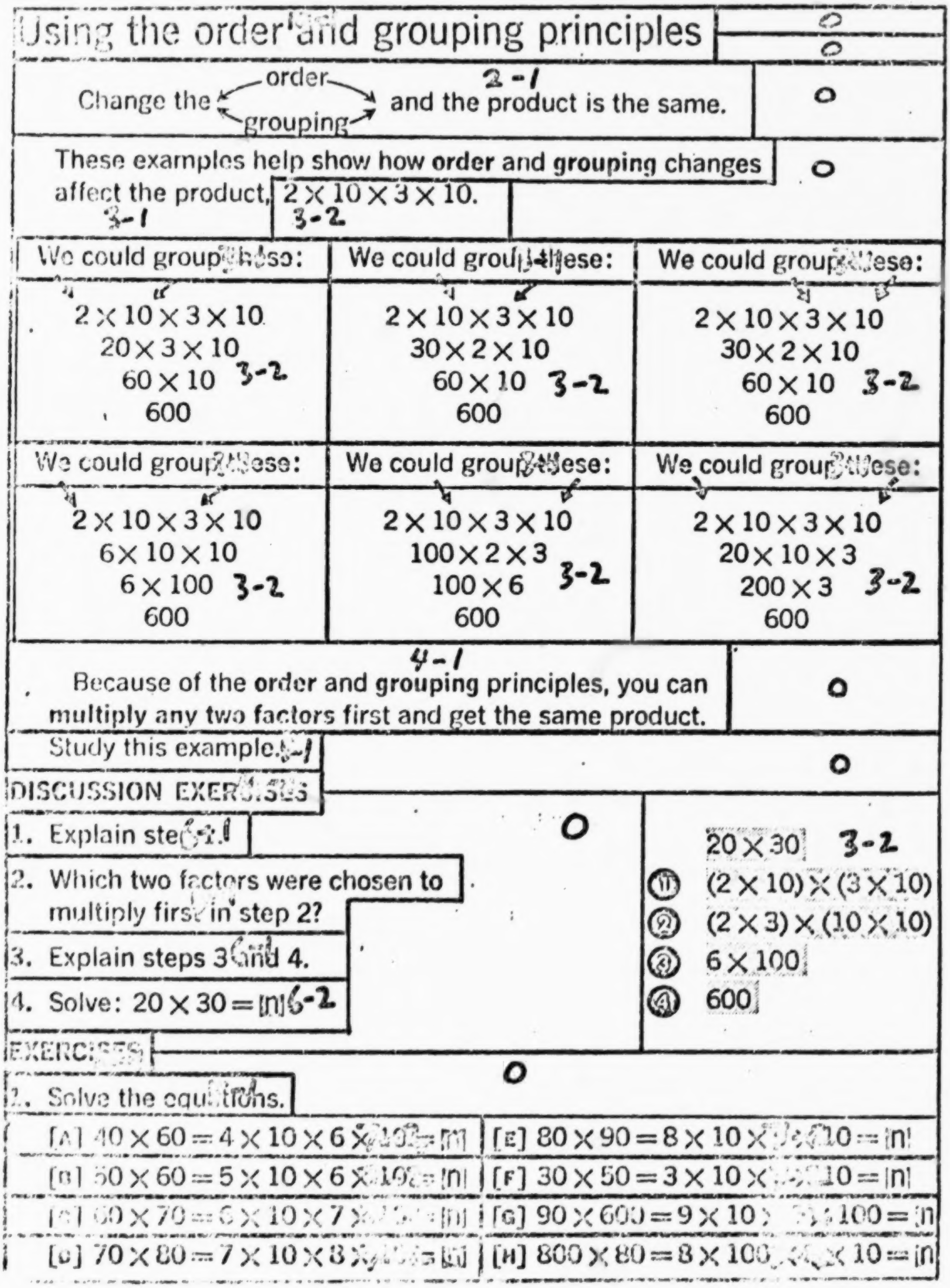
Eicholz and O'Daffer, Addison-Wesley
Publishing Company, 1968.)

Key to Practice Passage 1



Using the order and grouping principles

0

0

2-1

Change the order / grouping and the product is the same.

0

These examples help show how order and grouping changes affect the product, 2 × 10 × 3 × 10.

3-1 3-2

0

We could group these:	We could group these:	We could group these:
2 × 10 × 3 × 10 20 × 3 × 10 60 × 10 3-2 600	2 × 10 × 3 × 10 30 × 2 × 10 60 × 10 3-2 600	2 × 10 × 3 × 10 30 × 2 × 10 60 × 10 3-2 600
We could group these:	**We could group these:**	**We could group these:**
2 × 10 × 3 × 10 6 × 10 × 10 6 × 100 3-2 600	2 × 10 × 3 × 10 100 × 2 × 3 100 × 6 3-2 600	2 × 10 × 3 × 10 20 × 10 × 3 200 × 3 3-2 600

4-1

Because of the order and grouping principles, you can multiply any two factors first and get the same product.

0

Study this example. 5-1

0

DISCUSSION EXERCISES

0

1. Explain step 1.
2. Which two factors were chosen to multiply first in step 2?
3. Explain steps 3 and 4.
4. Solve: 20 × 30 = n 6-2

20 × 30 3-2

① (2 × 10) × (3 × 10)

② (2 × 3) × (10 × 10)

③ 6 × 100

④ 600

EXERCISES

0

1. Solve the equations.

[A] 40 × 60 = 4 × 10 × 6 × 10 = n	[E] 80 × 90 = 8 × 10 × [illegible] 10 = n
[B] 50 × 60 = 5 × 10 × 6 × 10 = n	[F] 30 × 50 = 3 × 10 × [illegible] 10 = n
[C] 60 × 70 = 6 × 10 × 7 × [illegible] = n	[G] 90 × 600 = 9 × 10 [illegible] 100 = n
[D] 70 × 80 = 7 × 10 × 8 × [illegible] = n	[H] 800 × 80 = 8 × 100 [illegible] × 10 = n

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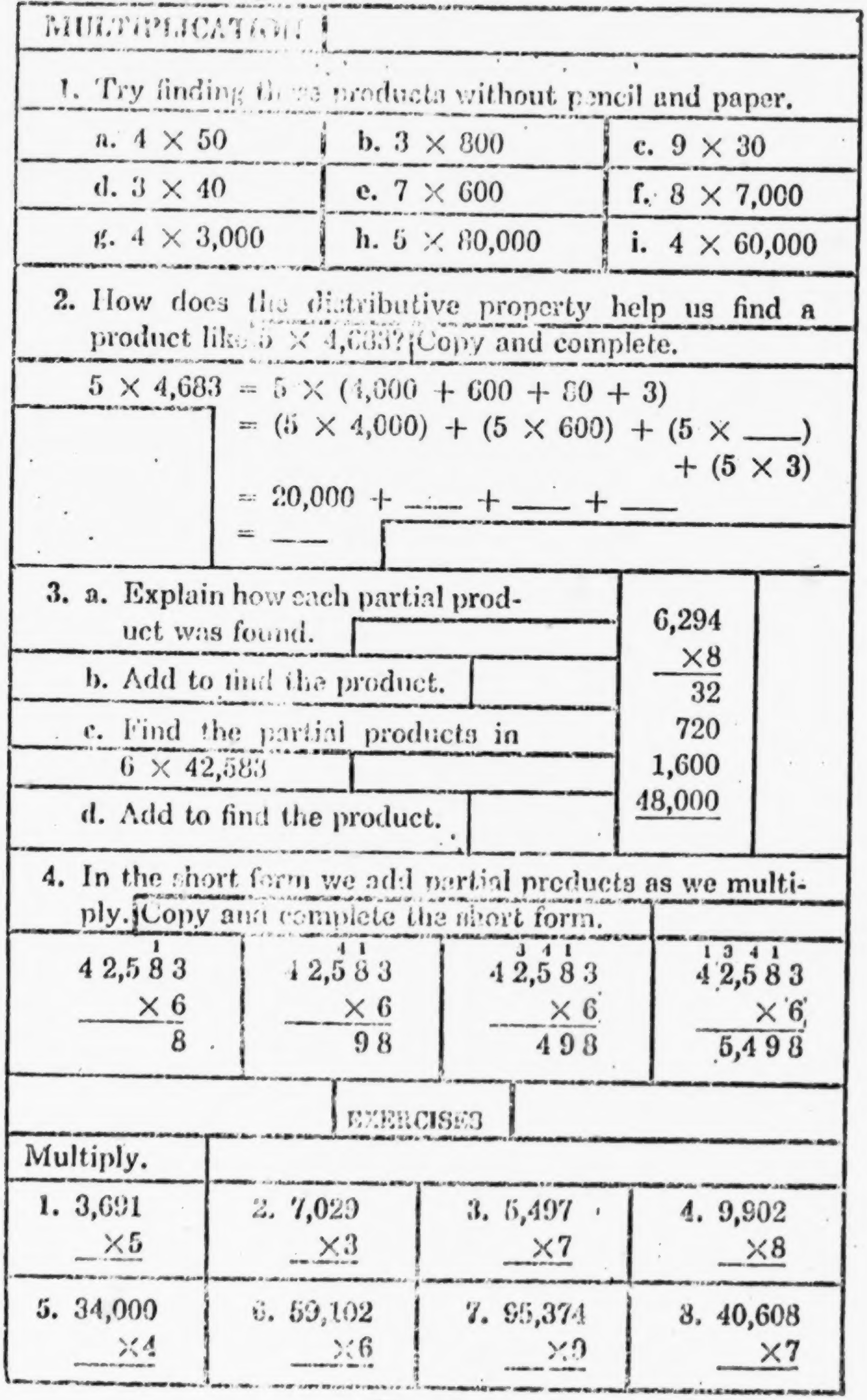
Practice Passage 2



MULTIPLICATION

1. Try finding these products without pencil and paper.

a. 4 × 50	b. 3 × 800	c. 9 × 30
d. 3 × 40	e. 7 × 600	f. 8 × 7,000
g. 4 × 3,000	h. 5 × 80,000	i. 4 × 60,000

2. How does the distributive property help us find a product like 5 × 4,683? Copy and complete.

$$
\begin{aligned}
5 \times 4{,}683 &= 5 \times (4{,}000 + 600 + 80 + 3)\\
&= (5 \times 4{,}000) + (5 \times 600) + (5 \times \_\_\_)\\
&\qquad + (5 \times 3)\\
&= 20{,}000 + \_\_\_ + \_\_\_ + \_\_\_\\
&= \_\_\_
\end{aligned}
$$

3. a. Explain how each partial product was found.

b. Add to find the product.

c. Find the partial products in 6 × 42,583

d. Add to find the product.

$$
\begin{array}{r}
6{,}294\\
\times 8\\
\hline
32\\
720\\
1{,}600\\
\underline{48{,}000}
\end{array}
$$

4. In the short form we add partial products as we multiply. Copy and complete the short form.

$$
\begin{array}{r}
{}^{1}\\
42{,}583\\
\times 6\\
\hline
8
\end{array}
\qquad
\begin{array}{r}
{}^{4\ 1}\\
42{,}583\\
\times 6\\
\hline
98
\end{array}
\qquad
\begin{array}{r}
{}^{3\ 4\ 1}\\
42{,}583\\
\times 6\\
\hline
498
\end{array}
\qquad
\begin{array}{r}
{}^{1\ 3\ 4\ 1}\\
42{,}583\\
\times 6\\
\hline
5{,}498
\end{array}
$$

EXERCISES

Multiply.

1. 3,691 ×5	2. 7,029 ×3	3. 5,497 ×7	4. 9,902 ×8
5. 34,000 ×4	6. 59,102 ×6	7. 95,374 ×9	8. 40,608 ×7

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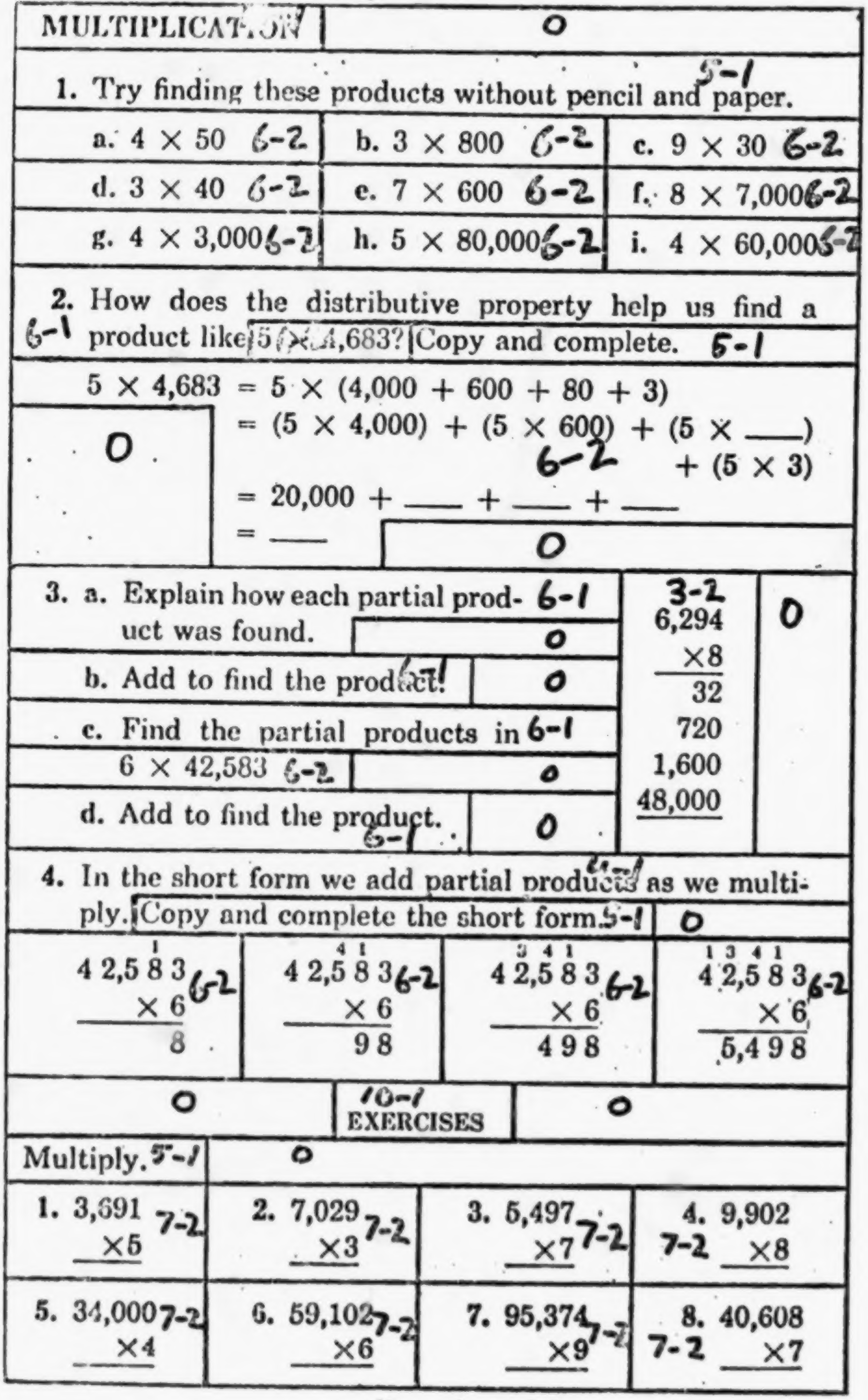
Keedy, et al., Holt, Rinehart and
Winston, Inc., 1970.)

Key to Practice Passage 2



MULTIPLICATION | 0

1. Try finding these products without pencil and paper. 5-1

a. 4 × 50 6-2	b. 3 × 800 6-2	c. 9 × 30 6-2
d. 3 × 40 6-2	e. 7 × 600 6-2	f. 8 × 7,000 6-2
g. 4 × 3,000 6-2	h. 5 × 80,000 6-2	i. 4 × 60,000 6-2

2. How does the distributive property help us find a product like 5 × 4,683? 6-1 Copy and complete. 5-1

0

$$5 \times 4{,}683 = 5 \times (4{,}000 + 600 + 80 + 3)$$
$$= (5 \times 4{,}000) + (5 \times 600) + (5 \times \underline{\quad}) + (5 \times 3)$$
$$= 20{,}000 + \underline{\quad} + \underline{\quad} + \underline{\quad}$$
$$= \underline{\quad}$$

6-2

0

3. a. Explain how each partial product was found. 6-1 | 0

b. Add to find the product. 6-1 | 0

c. Find the partial products in 6 × 42,583 6-1 6-2 | 0

d. Add to find the product. 6-1 | 0

3-2

$$\begin{array}{r} 6{,}294 \\ \times 8 \\ \hline 32 \\ 720 \\ 1{,}600 \\ \underline{48{,}000} \end{array}$$

0

4. In the short form we add partial products as we multiply. 6-1 Copy and complete the short form. 5-1 | 0

$\begin{array}{r} {}^{1} \\ 42{,}583 \\ \times 6 \\ \hline 8 \end{array}$ 6-2	$\begin{array}{r} {}^{4\ 1} \\ 42{,}583 \\ \times 6 \\ \hline 98 \end{array}$ 6-2	$\begin{array}{r} {}^{3\ 4\ 1} \\ 42{,}583 \\ \times 6 \\ \hline 498 \end{array}$ 6-2	$\begin{array}{r} {}^{1\ 3\ 4\ 1} \\ 42{,}583 \\ \times 6 \\ \hline 5{,}498 \end{array}$ 6-2

0 | 10-1 **EXERCISES** | 0

Multiply. 5-1 | 0

1. 3,691 ×5 7-2	2. 7,029 ×3 7-2	3. 5,497 ×7 7-2	4. 9,902 ×8 7-2
5. 34,000 ×4 7-2	6. 59,102 ×6 7-2	7. 95,374 ×9 7-2	8. 40,608 ×7 7-2

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Practice Passage 3

Using Equations and Inequalities

A 12 foot board is cut into 3 pieces of equal length. To find the length of each piece, consider the equation

$$3 \cdot x = 12 .$$

Solving this equation we find, $x = 4$. Hence, each piece is 4 feet long.

Five girls want to share 23 apples evenly. To find the greatest number of whole apples each girl can get, consider the inequality

$$n \cdot 5 < 23 .$$

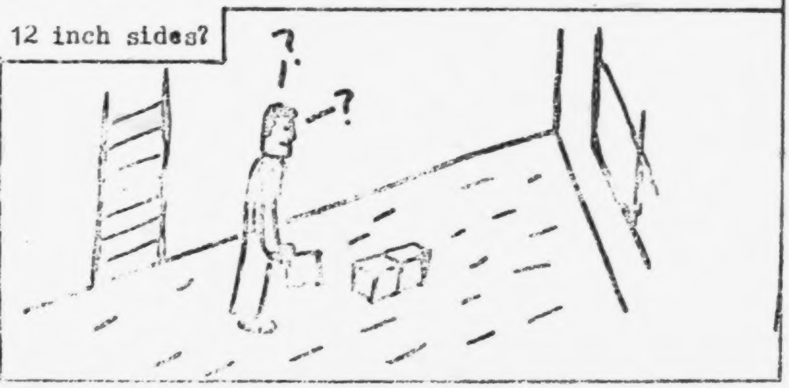
Using the replacement set of whole numbers, the solution set is $\{0, 1, 2, 3, 4\}$. So the greatest number of apples each girl can get is 4.

Exercises

Write and solve equations or inequalities to answer the problems.

1. In 3 days the liner on which Carol was traveling steamed 1536 miles. If it steamed the same distance each day then how far did it go each day?

2. How many tile would it take to cover the floor of a rectangular room with length 20 feet and width 13 feet if the tile are square with 12 inch sides?



Key to Practice Passage 3

Using Equations and Inequalities 10-1
0

A 12 foot board is cut into 3 pieces of equal length. 3-1
To find

the length of each piece, consider the equation 3-1
0

0
3-2 $3 \cdot x = 12$.
0

Solving this equation 3-1
we find, 3-2 $x = 4$.
Hence, each 3-1 piece is 4 feet

long. 3-1
0

Five girls want to 3 share 23 apples evenly.
To find the greatest 3-1

number of whole apples each girl can get, consider the inequality

0
3-2 $n \cdot 5 < 23$.
0

Using the replacement set of whole numbers, the solution set is 3-1

$\{0, 1, 2, 3, 4\}$ 3-2 .
So the greatest number of apples each girl 3-1

can get is 4. 3-1
0

Exercises 10-1
0

Write and solve equations or inequalities to answer the problems. 5-1

1. In 3 days the liner on which Carol was traveling steamed 1536

8-1 miles.
If it steamed the same distance each day then how far

8-1 did it go each day?
0

10-6

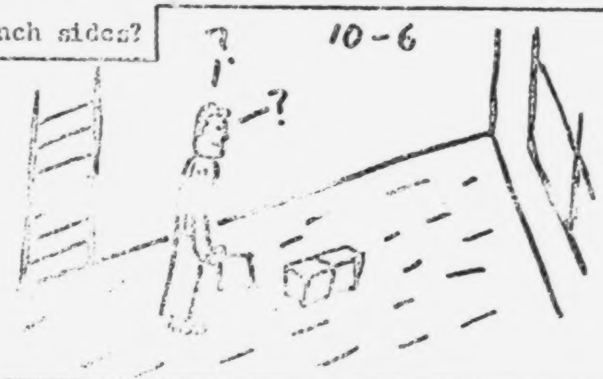
2. How many tile would it take to cover the floor of a rectangular

8-1 room with length 20 feet and width 13 feet if the tile are

square with 12 inch sides?

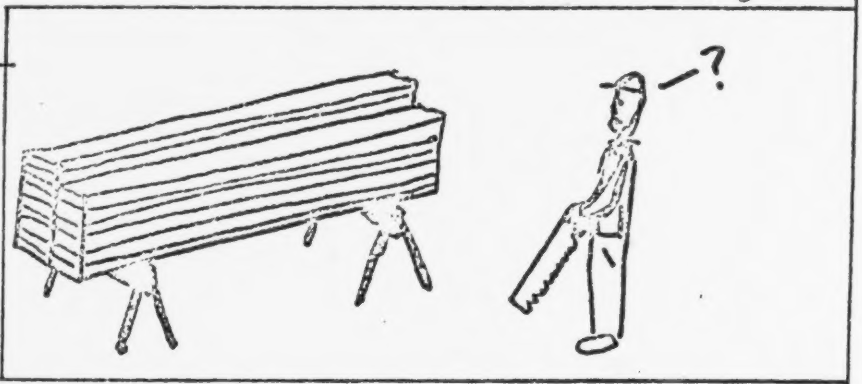
10-6

0



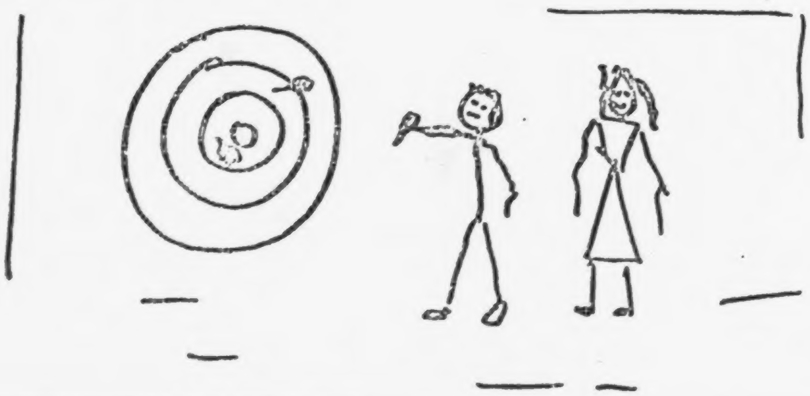
Practice Passage 3 (cont.)

3. How many 5 foot boards can be made out of 17 boards of length 12 feet?



4. If you bought 25 hotdogs for a cook-out and each person at the cook-out has to have 3 hotdogs, what is the greatest number of people you could have at the cook-out?

5. In a dart game Sam scored 25, 10, 25, and 35. Penny scored 25, 25, 15, and 20. Who won the game and by how much?



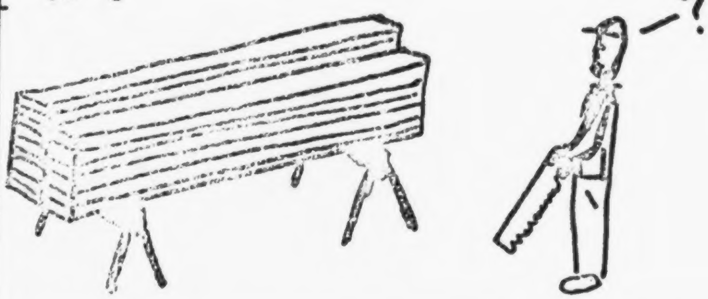
Key to Practice Passage 3 (cont.)

3. How many 5 foot boards can be made out of 17 boards of length

8-1 12 feet?

0

10-6



4. If you bought 25 hotdogs for a cook-out and each person at the

8-1 cook-out has to have 3 hotdogs, what is the greatest number of

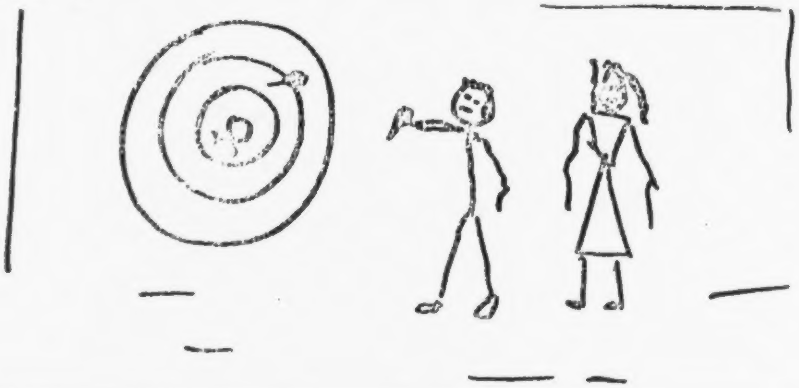
people you could have at the cook-out? 0

10-6

5. 8-1 In a dart game Sam scored 25, 10, 25, and 35. Penny 8-1 scored 25,

8-2 25, 15, and 20. Who won the 8-1 game and by how much? 0

10-6



Practice Passage 4

Using Equations and Inequalities

Equations and inequalities can be used to solve real world problems. To do this follow the steps outlined below.

1. Express the unknown quantity or what you want to find as some symbol, say x.
2. Using the information given write an equation such as

$$a + x = b$$

or an inequality such as

$$x \cdot c < d$$

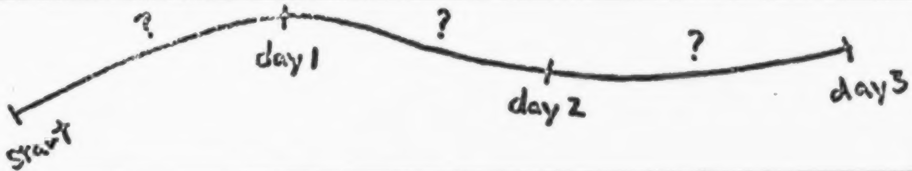
relating x to known quantities a, b, c, and d.

3. Solve the equation or inequality to find the values of x and thus the answer to the problem.

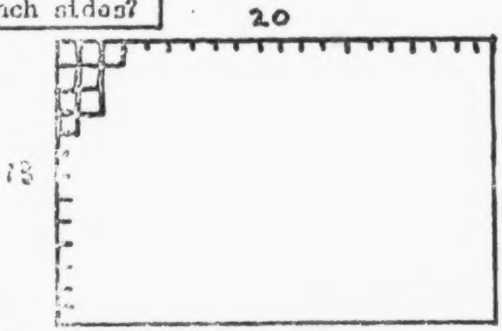
Exercises

Write and solve equations or inequalities to answer the problems.

1. In 3 days the liner on which Carol was traveling steamed 1536 miles. If it steamed the same distance each day how far did it go each day?



2. How many tile would it take to cover the floor of a rectangular room with length 20 feet and width 13 feet if the tile are square with 12 inch sides?



Key to Practice Passage 4

Using Equations and Inequalities 10-1 | 0

Equations and inequalities can be used 4-1 to solve real world problems.

To do this follow the steps outlined 4-1 below. | 0

1. Express the unknown quantity or what you want to find as
some symbol, say x. 4-1 | 0

2. Using the information 4-1 given write an equation such as

0 | $a + x = b$ 4-2 | 0

or an inequality 4-1 such as

0 | $x \cdot c < d$ 4-2

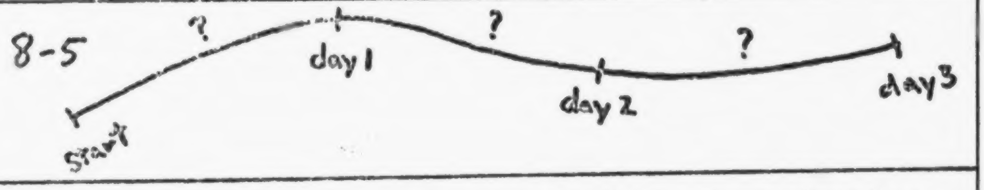
relating x 4-1 to known quantities | a, b, c, 4-2 and d.

3. Solve the equation or inequality to find the values of x
and thus the answer 4-1 to the problem. | 0

Exercises 10-1 | 0

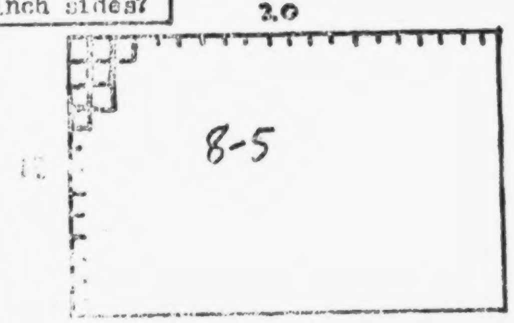
Write and solve equations 5-1 or inequalities to answer the problems.

1. In 3 days the liner on which Carol was traveling steamed 1536
8-1 miles. | If it steamed the same distance each day how far did
8-1 it go each day? | 0



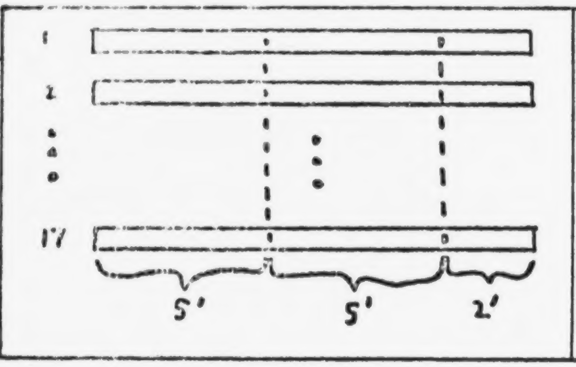
2. How many tile would it take to cover the floor of a rectangular
8-1 room with length 20 feet and width 13 feet if the tile are
square with 12 inch sides? | 0

0

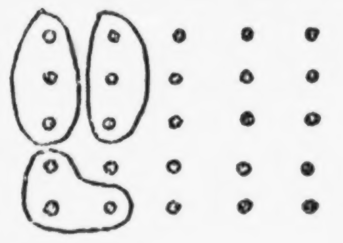


Practice Passage 4 (cont.)

3. How many 5 foot boards can be made out of 17 boards of length 12 feet?



4. If you bought 25 hotdogs for a cook-out and each person at the cook-out has to have 3 hotdogs, what is the greatest number of people you could have at the cook-out?



5. In a dart game Sam scored 25, 10, 25, and 35. Penny scored 25, 25, 15, and 20. Who won the game and by how much?

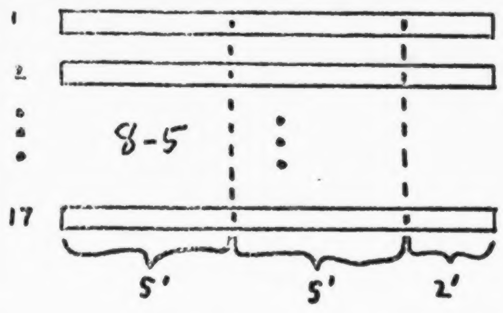
Sam	Penny
25	25
10	25
25	15
35	20

Key to Practice Passage 4 (cont.)

3. How many 5 foot boards can be made out of 17 boards of length

8-1 12 feet?

0



8-5

0

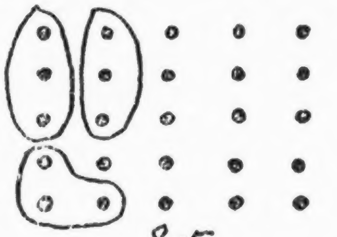
4. If you bought 25 hotdogs for a cook-out and each person at the

8-1 cook-out has to have 3 hotdogs, what is the greatest number of

people you could have at the cook-out?

0

0



8-5

0

5. 8-1 In a dart game Sam scored 25, 10, 8-2 25, and 35. Penny 8-1 scored 25,

8-2 25, 15, and 20. Who won the game and 8-1 by how much? 0

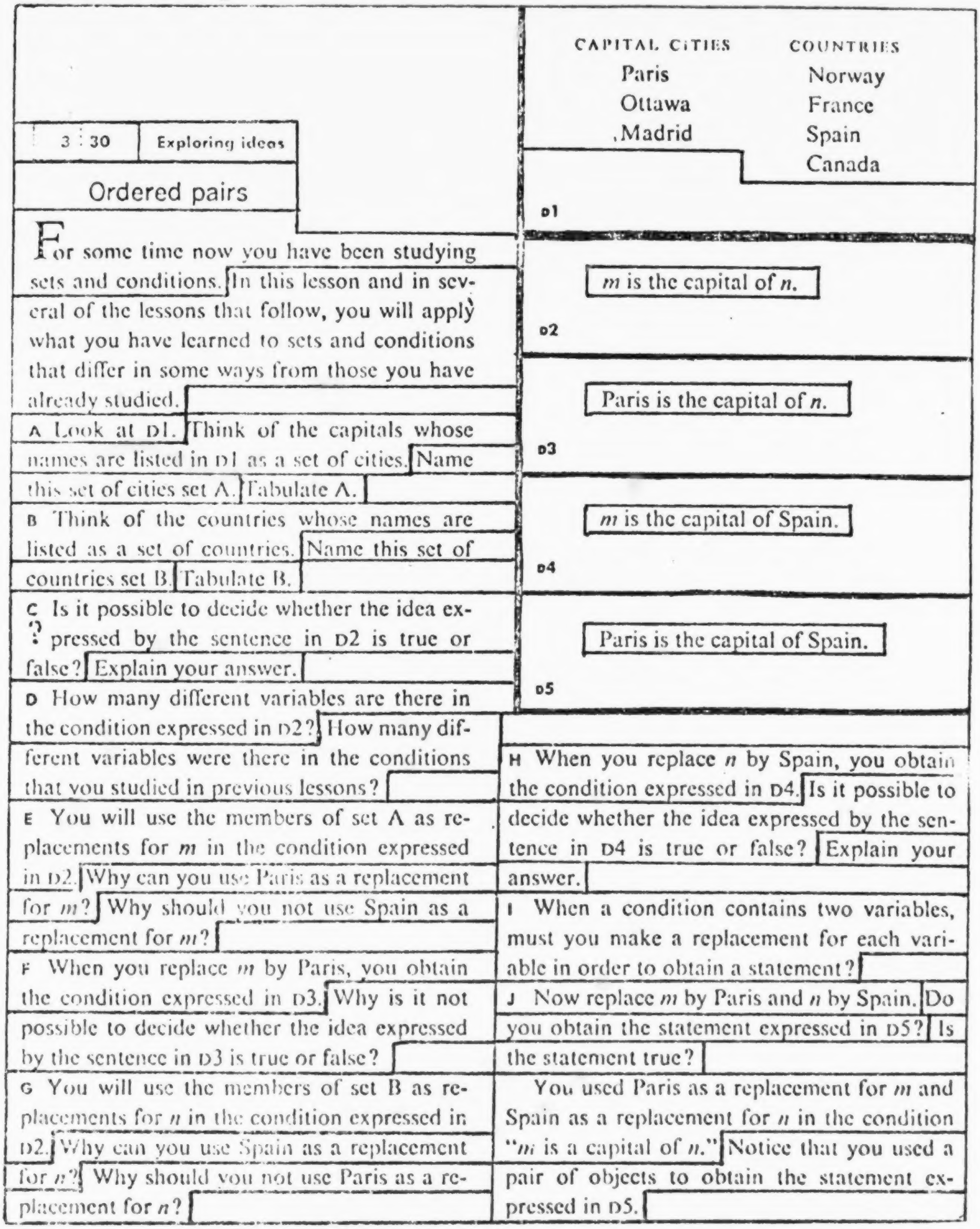
0

Sam	Penny
25	25
10	25
25	15
35	20

8-7

0

Practice Passage 5



3 : 30 Exploring ideas

Ordered pairs

For some time now you have been studying sets and conditions. In this lesson and in several of the lessons that follow, you will apply what you have learned to sets and conditions that differ in some ways from those you have already studied.

A Look at D1. Think of the capitals whose names are listed in D1 as a set of cities. Name this set of cities set A. Tabulate A.

B Think of the countries whose names are listed as a set of countries. Name this set of countries set B. Tabulate B.

C Is it possible to decide whether the idea expressed by the sentence in D2 is true or false? Explain your answer.

D How many different variables are there in the condition expressed in D2? How many different variables were there in the conditions that you studied in previous lessons?

E You will use the members of set A as replacements for *m* in the condition expressed in D2. Why can you use Paris as a replacement for *m*? Why should you not use Spain as a replacement for *m*?

F When you replace *m* by Paris, you obtain the condition expressed in D3. Why is it not possible to decide whether the idea expressed by the sentence in D3 is true or false?

G You will use the members of set B as replacements for *n* in the condition expressed in D2. Why can you use Spain as a replacement for *n*? Why should you not use Paris as a replacement for *n*?

CAPITAL CITIES	COUNTRIES
Paris	Norway
Ottawa	France
Madrid	Spain
	Canada

D1

m is the capital of *n*.

D2

Paris is the capital of *n*.

D3

m is the capital of Spain.

D4

Paris is the capital of Spain.

D5

H When you replace *n* by Spain, you obtain the condition expressed in D4. Is it possible to decide whether the idea expressed by the sentence in D4 is true or false? Explain your answer.

I When a condition contains two variables, must you make a replacement for each variable in order to obtain a statement?

J Now replace *m* by Paris and *n* by Spain. Do you obtain the statement expressed in D5? Is the statement true?

You used Paris as a replacement for *m* and Spain as a replacement for *n* in the condition "*m* is a capital of *n*." Notice that you used a pair of objects to obtain the statement expressed in D5.

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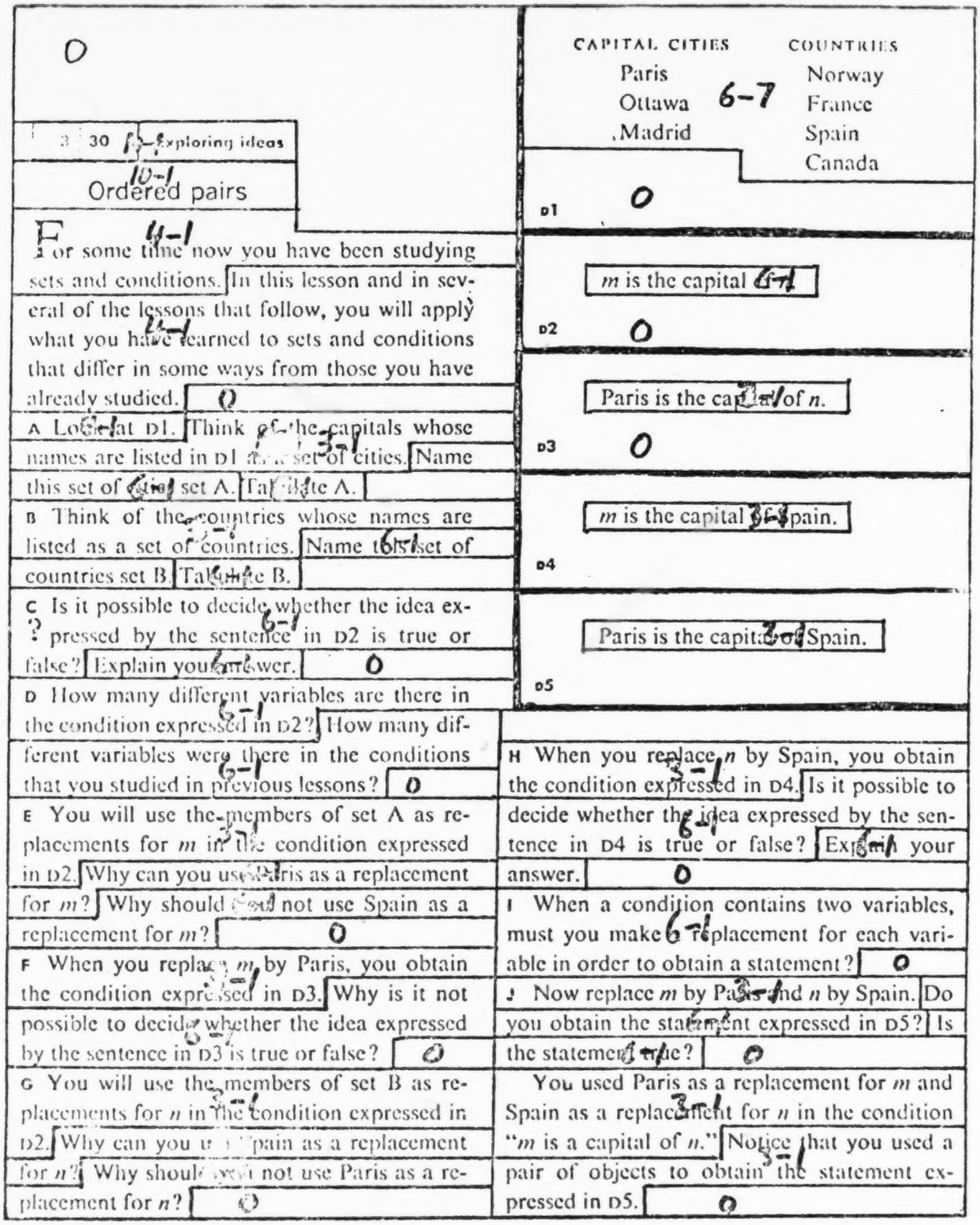
Seeing Through Mathematics, Book 1,
VanEngen et al., Scott, Forsman and
Company, 1962.)

Key to Practice Passage 5



0

30 Exploring ideas

10-1

Ordered pairs

4-1

For some time now you have been studying sets and conditions. In this lesson and in several of the lessons that follow, you will apply what you have learned to sets and conditions that differ in some ways from those you have already studied. 0

2-1

A Look at D1. Think of the capitals whose names are listed in D1 as a set of cities. Name this set of cities set A. Tabulate A.

6-1 3-1 6-1

B Think of the countries whose names are listed as a set of countries. Name this set of countries set B. Tabulate B.

6-1 6-1

C Is it possible to decide whether the idea expressed by the sentence in D2 is true or false? Explain your answer. 0

6-1 6-1

D How many different variables are there in the condition expressed in D2? How many different variables were there in the conditions that you studied in previous lessons? 0

6-1 6-1

E You will use the members of set A as replacements for m in the condition expressed in D2. Why can you use Paris as a replacement for m? Why should you not use Spain as a replacement for m? 0

3-1

F When you replace m by Paris, you obtain the condition expressed in D3. Why is it not possible to decide whether the idea expressed by the sentence in D3 is true or false? 0

3-1 6-1

G You will use the members of set B as replacements for n in the condition expressed in D2. Why can you use Spain as a replacement for n? Why should you not use Paris as a replacement for n? 0

3-1

6-7

CAPITAL CITIES	COUNTRIES
Paris	Norway
Ottawa	France
Madrid	Spain
	Canada

D1 0

m is the capital 6-1

D2 0

Paris is the capital of n. 3-1

D3 0

m is the capital of Spain. 3-1

D4

Paris is the capital of Spain. 3-1

D5

H When you replace n by Spain, you obtain the condition expressed in D4. Is it possible to decide whether the idea expressed by the sentence in D4 is true or false? Explain your answer. 0

3-1 6-1 6-1

I When a condition contains two variables, must you make a replacement for each variable in order to obtain a statement? 0

6-1

J Now replace m by Paris and n by Spain. Do you obtain the statement expressed in D5? Is the statement true? 0

3-1 6-1 3-1

You used Paris as a replacement for m and Spain as a replacement for n in the condition "m is a capital of n." Notice that you used a pair of objects to obtain the statement expressed in D5. 0

3-1 3-1

Practice Passage 5 (cont.)

K The first object of the pair, Paris, is the *first component* of the pair. The first component is a member of what set?

L The second object of the pair, Spain, is the *second component* of the pair. The second component is a member of what set?

In the exercises above, you used a city for the first component and a country for the second component of a pair of objects. A pair of objects whose components occur in a special order is an *ordered pair.*

M Look at D6. Notice that the names of the components of the ordered pair Paris, Spain are written within parentheses. D6 also shows how to read and write the name of this ordered pair.

N Think about the condition "*m* is the capital of *n*." When you replace *m* by a member of {Paris, Ottawa, Madrid} and *n* by a member of {Norway, France, Spain, Canada}, do you obtain a statement? Do the sentences in D7 express all the statements you can obtain?

O For each statement expressed in D7, write the name of the ordered pair that was used to make replacements. Which do you write first, the name of the replacement for *m* or the name of the replacement for *n*?

P Which of the statements expressed in D7 are true?

Q What ordered pairs were used to obtain true statements?

R Read the symbols (Paris, Norway) and {Paris, Norway}. How do the ideas expressed by these symbols differ?

Now you will study some conditions that concern numbers. You will also learn about ordered pairs whose components are numbers.

A Read the open sentence in D8. What are the variables in the condition expressed by the [illegible]

"The ordered pair

(Paris, Spain)

whose first component
is Paris and

whose second component
is Spain"

D6

- **A** Paris is the capital of Norway.
- **B** Paris is the capital of France.
- **C** Paris is the capital of Spain.
- **D** Paris is the capital of Canada.
- **E** Ottawa is the capital of Norway.
- **F** Ottawa is the capital of France.
- **G** Ottawa is the capital of Spain.
- **H** Ottawa is the capital of Canada.
- **I** Madrid is the capital of Norway.
- **J** Madrid is the capital of France.
- **K** Madrid is the capital of Spain.
- **L** Madrid is the capital of Canada.

D7

$x < y.$
$A = \{1, 2, 3, 4\}.$
$B = \{2, 4, 6\}.$

D8

ordered pair (ôr′dərd pâr). A pair of objects in which the objects occur in a special order. The object that occurs first is the **first component** (kəm pō′nənt) of the ordered pair. The object that occurs second is the **second component** of the ordered pair.

B Also read the tabulations of sets A and B in D8. You will use the members of set A as re- placements for x in $x < y$. You will use the members of set B as replacements for y in $x < y$.

Key to Practice Passage 5 (cont.)

0

K The first object, of the pair, Paris, is the *first component* of the pair. The first component is a member of what set? 0

1-1

L The second object of the pair, Spain, is the *second component* of the pair. The second component is a member of what set? 0

1-1

In the exercises above, you used a city for the first component and a country for the second component of a pair of objects. A pair of objects whose components occur in a special order is an *ordered pair*. 0

1-1

M Look at D6. Notice that the names of the components of the ordered pair Paris, Spain are written within parentheses. D6 also shows how to read and write the name of this ordered pair. 0

3-1

N Think about the condition "*m* is the capital of *n*." When you replace *m* by a member of {Paris, Ottawa, Madrid} and *n* by a member of {Norway, France, Spain, Canada}, do you obtain a statement? Do the sentences in D7 express all the statements you can obtain?

5-1

O For each statement expressed in D7, write the name of the ordered pair that was used to make replacements. Which do you write first, the name of the replacement for *m* or the name of the replacement for *n*? 0

6-1

6-1

P Which of the statements expressed in D7 are true? 0

Q What ordered pairs were used to obtain true statements? 0

R Read the symbols (Paris, Norway) and ? (Paris, Norway). How do the ideas expressed by these symbols differ? 0

6-1

Now you will study some conditions that concern numbers. You will also learn about ordered pairs whose components are numbers.

6-1

A Read the open sentence in D8. What are the variables in the condition expressed by the sentence? 0

"The ordered pair 0

0

(Paris, Spain)

3-1

whose first component
is Paris and

whose second component
is Spain" 0

0

D6

0

A Paris is the capital of Norway.
B Paris is the capital of France.
C Paris is the capital of Spain.
D Paris is the capital of Canada.
E Ottawa is the capital of Norway.
F Ottawa is the capital of France.
G Ottawa is the capital of Spain.
H Ottawa is the capital of Canada.
I Madrid is the capital of Norway.
J Madrid is the capital of France.
K Madrid is the capital of Spain.
L Madrid is the capital of Canada.

6-1

D7 0

$x < y.$
$A = \{1, 2, 3, 4\}.$
$B = \{2, 4, 6\}.$

6-2

D8 0

1-1

ordered pair (ôr′dərd pâr). A pair of objects in which the objects occur in a special order. The object that occurs first is the **first component** (kəm pō′nənt) of the ordered pair. The object that occurs second is the **second component** of the ordered pair. 0

1-1

1-1

0

B Also read the tabulations of sets A and B in D8. You will use the members of set A as replacements for x in $x < y$. You will use the members of set B as replacements for y in $x < y$. 0

3-1

3-1

Practice Passage 5 (cont.)

C To obtain a statement from a condition in two variables, you must make replacements for how many variables?

D Do the sentences in D9 express all the statements that you can obtain from $x < y$ when you use the members of set A and set B, tabulated in D8, as replacements for x and y?

E Think of the statement expressed by the first sentence in D9. (1, 2) was used in making replacements to obtain this statement. D10 shows a short and convenient way to read and write the name of this ordered pair.

F Select four of the statements expressed in D9. For each statement, name the ordered pair that was used in making replacements for x and y.

G Which statements expressed in D9 are true? What ordered pairs were used to obtain true statements?

H The sentences $x + x = 4$ and $x + y = 4$ express two conditions. How many different variables are there in each condition? How do the two conditions differ?

In this lesson you have learned that ordered pairs are used to make replacements for two variables.

On your own

Write a name for each of the ordered pairs described in exercises 1 through 5.

1 The ordered pair whose first component is Mark Twain and whose second component is *Tom Sawyer*

2 The ordered pair whose first component is Mary and whose second component is guitar

3 The ordered pair whose first component is 15° and whose second component is 8 P.M.

4 The ordered pair whose first component is Larry and whose second component is 141

5 The ordered pair whose first component is 1 and whose second component is 13

$x < y$.
$A = \{1, 2, 3, 4\}$.
$B = \{2, 4, 6\}$.

D8

$1 < 2$.	$2 < 2$.	$3 < 2$.	$4 < 2$.
$1 < 4$.	$2 < 4$.	$3 < 4$.	$4 < 4$.
$1 < 6$.	$2 < 6$.	$3 < 6$.	$4 < 6$.

D9

"The ordered pair

(1, 2)

one, two"

D10

Four conditions are expressed in exercises 6 through 9. Use the members of set C, tabulated below, as replacements for x. Use the members of set D, tabulated below, as replacements for y. For each exercise write sentences to express all the statements that you can obtain from the condition. Then write "T" after each sentence that expresses a true statement.

$C = \{2, 4, 6\}$.	$D = \{5, 10, 15\}$.
6 $6 + x = y$.	8 $x + 4 = y$.
7 $x + 5 < y$.	9 $y > x + 1$.

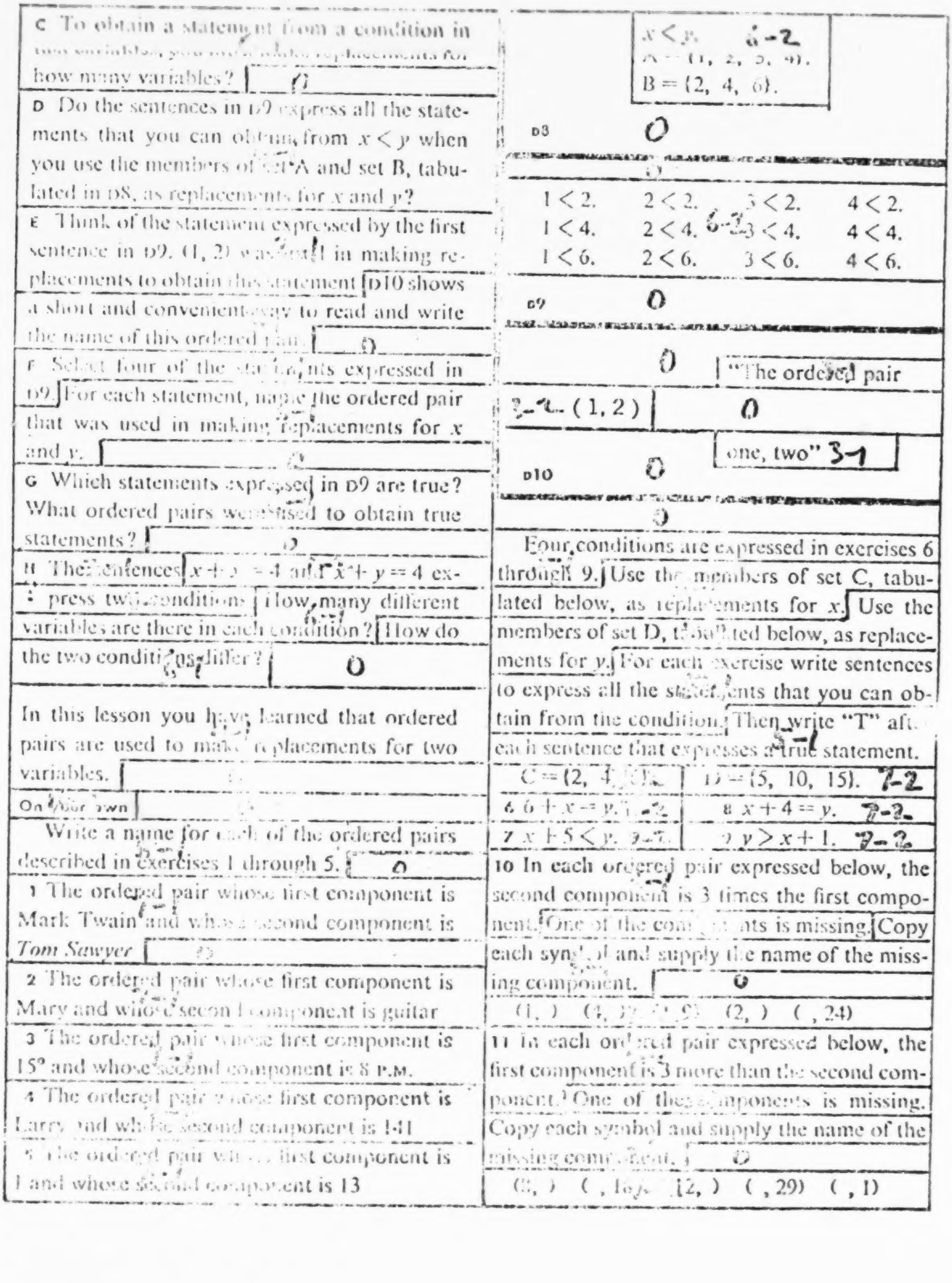
10 In each ordered pair expressed below, the second component is 3 times the first component. One of the components is missing. Copy each symbol and supply the name of the missing component.

(1,) (4,) (, 9) (2,) (, 24)

11 In each ordered pair expressed below, the first component is 3 more than the second component. One of the components is missing. Copy each symbol and supply the name of the missing component.

(8,) (, 18) (12,) (, 29) (, 1)

Key to Practice Passage 5 (cont.)



C To obtain a statement from a condition in two variables, you must make replacements for how many variables? 0

D Do the sentences in D9 express all the statements that you can obtain from $x < y$ when you use the members of set A and set B, tabulated in D8, as replacements for x and y?

E Think of the statement expressed by the first sentence in D9. (1, 2) was used in making replacements to obtain this statement. D10 shows a short and convenient way to read and write the name of this ordered pair. 0

F Select four of the statements expressed in D9. For each statement, name the ordered pair that was used in making replacements for x and y. 0

G Which statements expressed in D9 are true? What ordered pairs were used to obtain true statements? 0

H The sentences $x + y = 4$ and $x + y = 4$ express two conditions. How many different variables are there in each condition? How do the two conditions differ? 0

In this lesson you have learned that ordered pairs are used to make replacements for two variables.

On your own

Write a name for each of the ordered pairs described in exercises 1 through 5. 0

1 The ordered pair whose first component is Mark Twain and whose second component is *Tom Sawyer*

2 The ordered pair whose first component is Mary and whose second component is guitar

3 The ordered pair whose first component is 15° and whose second component is 8 P.M.

4 The ordered pair whose first component is Larry and whose second component is 141

5 The ordered pair whose first component is 1 and whose second component is 13

$x < y$. 6-2
A = {1, 2, 3, 4}.
B = {2, 4, 6}.

D8 0

0

$1 < 2$.	$2 < 2$.	$3 < 2$.	$4 < 2$.
$1 < 4$.	$2 < 4$. 6-2	$3 < 4$.	$4 < 4$.
$1 < 6$.	$2 < 6$.	$3 < 6$.	$4 < 6$.

D9 0

0 "The ordered pair

2-2 (1, 2) 0

one, two" 3-1

D10 0

0

Four conditions are expressed in exercises 6 through 9. Use the members of set C, tabulated below, as replacements for x. Use the members of set D, tabulated below, as replacements for y. For each exercise write sentences to express all the statements that you can obtain from the condition. Then write "T" after each sentence that expresses a true statement. 3-1

C = {2, 4, 6}. D = {5, 10, 15}. 7-2

6 $6 + x = y$. 7-2
8 $x + 4 = y$. 7-2
7 $x + 5 < y$. 7-2
9 $y > x + 1$. 7-2

10 In each ordered pair expressed below, the second component is 3 times the first component. One of the components is missing. Copy each symbol and supply the name of the missing component. 0

(1,) (4,) (, 9) (2,) (, 24)

11 In each ordered pair expressed below, the first component is 3 more than the second component. One of the components is missing. Copy each symbol and supply the name of the missing component. 0

(8,) (, 16) (12,) (, 29) (, 1)

Practice Passage 6

Graphing Linear Inequalities

The graph of $3x - 4y < 12$ is the set of points whose coordinates satisfy the inequality. Below is a method for graphing an inequality.

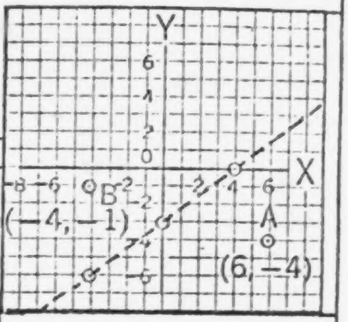
EXAMPLE

Graph $3x - 4y < 12$.

1 Graph the equation $3x - 4y = 12$.

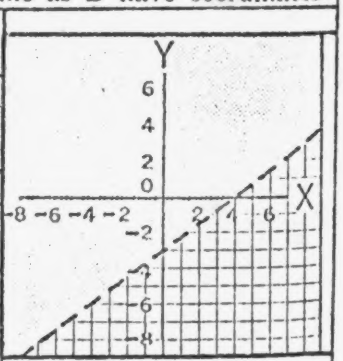
x	0	4	-4
y	-3	0	-6

Use a dashed line to represent the graph of the line, as shown at the right.



2 Points on one side of the line have coordinates that satisfy $3x - 4y < 12$. *Choose a point on one side of the line and see if its coordinates satisfy the inequality.* Do the coordinates of A satisfy $3x - 4y < 12$? Is $3(6) - 4(-4) < 12$? Is $18 + 16 < 12$? Then A is not on the graph of $3x - 4y < 12$. Do the coordinates of B satisfy $3x - 4y < 12$? Is $3(-4) - 4(-1) < 12$? Is $-12 + 4 < 12$? Then B is on the graph of $3x - 4y < 12$.

3 All points on the same side of the line as B have coordinates that satisfy $3x - 4y < 12$. *Shade this side of the line,* as shown at the right. (Dots are too difficult to do by hand.) The portion shown shaded represents the half-plane that is the graph of $3x - 4y < 12$. It extends indefinitely upward to the left. The dashed line is not part of the graph of the inequality. Why?



$3x - 4y < 12$ is a ***linear inequality*** because the boundary of its graph is a straight line.

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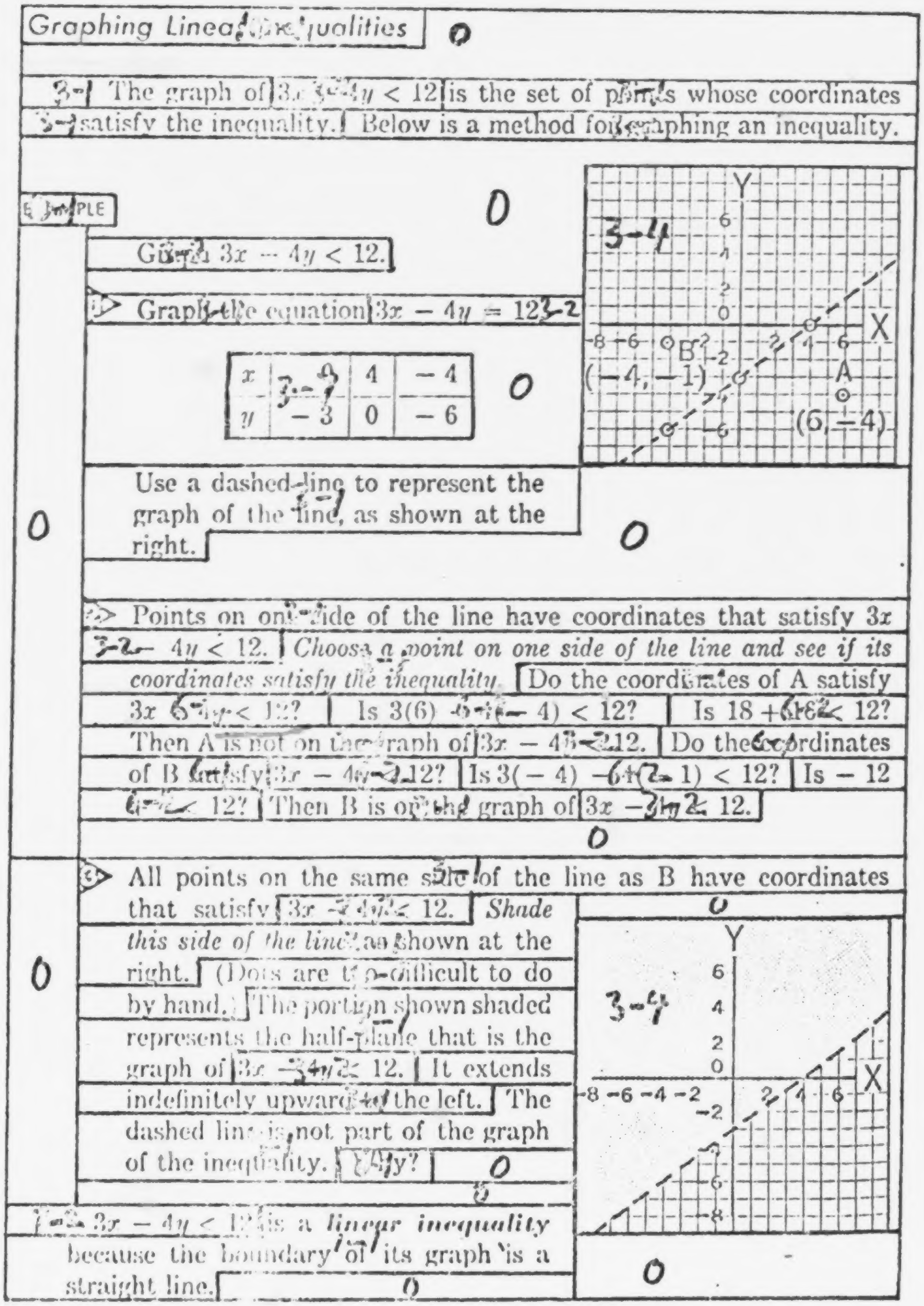
Contemporary Algebra, Book 1,
Smith et al., Harcourt, Brace and
World, 1962.)

Key to Practice Passage 6



Graphing Linear Inequalities

0

3-1 The graph of 3x − 4y < 12 is the set of points whose coordinates satisfy the inequality. Below is a method for graphing an inequality.

EXAMPLE

0

Given 3x − 4y < 12.

① Graph the equation 3x − 4y = 12. 3-2

x	0	4	−4
y	−3	0	−6

3-7

0

Use a dashed line to represent the graph of the line, as shown at the right.

0

0

3-4
Y
X
6
4
2
0
−8
−6
−2
2
4
6
−6
B
(−4, −1)
A
(6, −4)

② Points on one side of the line have coordinates that satisfy 3x − 4y < 12. *Choose a point on one side of the line and see if its coordinates satisfy the inequality.* Do the coordinates of A satisfy 3x − 4y < 12? Is 3(6) − 4(− 4) < 12? Is 18 + 16 < 12? Then A is not on the graph of 3x − 4y < 12. Do the coordinates of B satisfy 3x − 4y < 12? Is 3(− 4) − 4(− 1) < 12? Is − 12 + 4 < 12? Then B is on the graph of 3x − 4y < 12.

0

0

③ All points on the same side of the line as B have coordinates that satisfy 3x − 4y < 12. *Shade this side of the line,* as shown at the right. (Dots are too difficult to do by hand.) The portion shown shaded represents the half-plane that is the graph of 3x − 4y < 12. It extends indefinitely upward to the left. The dashed line is not part of the graph of the inequality. Why?

0

0

3x − 4y < 12 is a ***linear inequality*** because the boundary of its graph is a straight line.

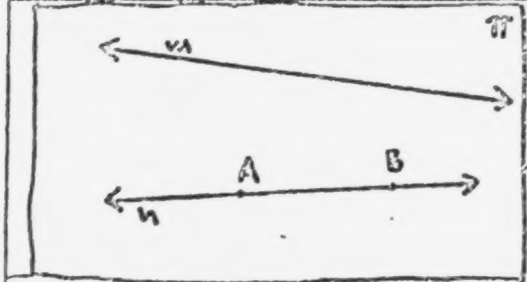
0

0

Practice Passage 7

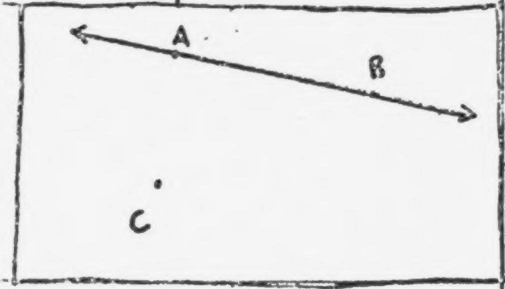
Some Logical Consequences of the Axioms

Theorem 1: If m is a line in plane π then there is a point in π which is not in m.



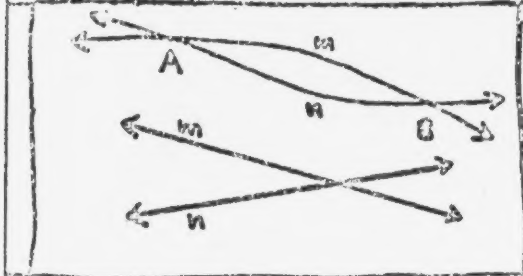
Suppose m is a line in plane π. By Axiom 1(a) there is a line n in π with $n \neq m$. By Axiom 1(b) there are distinct points A and B in line n. Now if both A and B were in m then we would have $m = n$ which is not the case. Hence, at least one of the points A or B is not in m. Thus, we have the above theorem.

Theorem 2: There are at least three non-collinear points in a given plane.



By Axiom 1(a) there exists a line m in a plane π. By Axiom 1(b) there are distinct points A and B on m. By Theorem 1 there exists a point C in plane π but not in line m. Hence, we have our second theorem.

Theorem 3: Two distinct lines can not have more than one point in common.

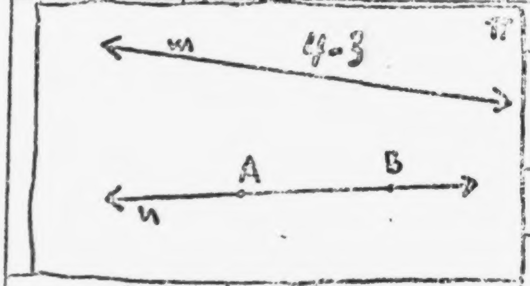


Let m and n be two distinct lines in a plane π. We know such lines exist by Axiom 1(a). Assume there are two distinct points A and B so that $A \in m \cap n$ and $B \in m \cap n$. Then m and n are two distinct lines each containing points A and B. But, this is impossible by Axiom 2. Hence, the above assumption is impossible and we have proven Theorem 3.

Key to Practice Passage 7

Some Logical Consequences of the axioms | 0

Theorem 1: 2-1 | If m is a line in plane π then there is a point in π which is not in m. 2-1 | 0 | 0



Suppose m is a line in plane π. 4-1

By Axiom 1(a) there is a line n in π 4-1

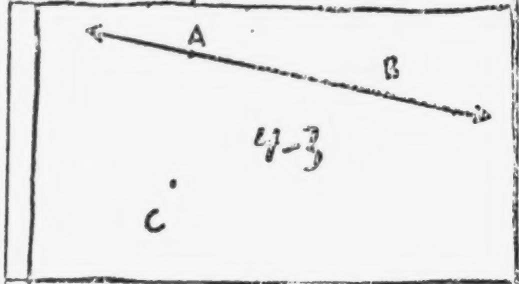
with | $n \neq m$. 4-2 | By Axiom 1(b) there are distinct points A and B in line n. 4-1

Now if both A and B were in m then 4-1

we would have 4-1 | $m = n$ 4-2 | which is not the case. 4-1 | Hence, at least one of the 4-1

points A or B is not in m. 4-1 | Thus, we have the above theorem. 4-1 | 0

Theorem 2: 2-1 | There are at least three non-collinear points in a given plane. 2-1

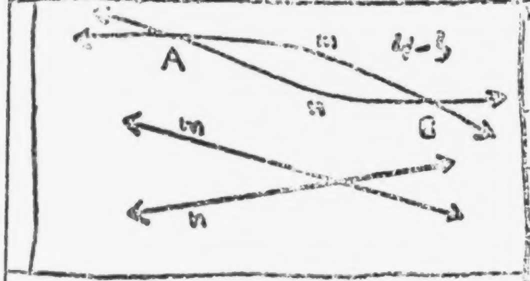


By Axiom 1(a) there exists a line m in a plane π. 4-1 | By Axiom 1(b) there are distinct points A and B on m. 4-1

By Theorem 1 there exists a point C in plane π but not in line m. 4-1 | Hence 4-1 we have our second theorem. 4-1 | 0

0

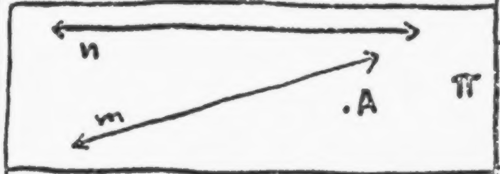
Theorem 3: 2-1 | Two distinct lines can not have more than one point in common. 2-1

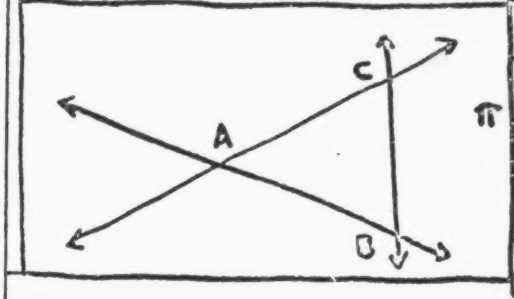


Let m and n be two distinct lines in a plane π. 4-1 | We know such lines exist by Axiom 1(a). 4-1 | Assume there are two distinct points 4-1 | A and B so that $A \in m \cap n$ 4-2 | and $B \in m \cap n$. 4-2 | Then m and n are two distinct lines each containing points A and B. 4-1 | But, this is impossible by Axiom 2. 4-1 | Hence, the above assumption is impossible and we have proven Theorem 3. 4-1 | 0

Practice Passage 7 (cont.)

Theorem 4: If A is a point in a plane π then there is a line in π which does not contain A.





Let A be any point in π. By Axiom 1(a) there are two distinct lines m and n in π. If either m or n does not contain A then the theorem is true. Suppose m and n both contain A. Then there exist points B and C with B in m, C in n, $B \neq A$, and $C \neq A$ by Axiom 1(b). Furthermore, $B \neq C$ by Theorem 3. Now, there is a line l containing B and C by Axiom 2. This line l can not contain A since if it did we would have A, B, and C all on l and l = m = n which is impossible. Thus, we have established the above theorem.

Exercises

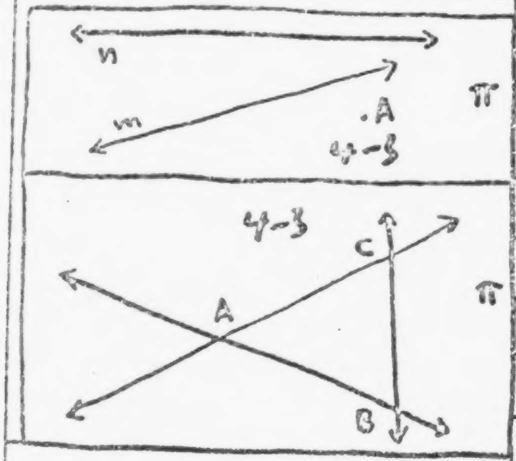
Prove the following theorems.

1. Theorem 5: If A is a point in plane π then there are at least two lines in π each containing point A.
2. Theorem 6: There are at least three non-concurrent lines in plane π.
3. Theorem 7: If each of two lines in π is parallel to the same line in π then they are parallel to each other.
4. Theorem 8: If m is any line in plane π then there are at least two points in π which are not in line m.
5. Theorem 9: If A is any point in plane π then there are at least two lines in π which do not contain A.

Key to Practice Passage 7 (cont.)

Theorem 4: If A is a point in a plane π then there is a line in π

0 — 2-1 which does not contain A. — 0



Let A be any 4-1 point in π. By Axiom 1(a) 4-1 there are two distinct lines m and n in π. If either m or n does not 4-1 contain A then the theorem is true.

Suppose m and n both contain A. Then 4-1 4-1 there exist 4-1 points B and C with B in 4-2 m, C in n, B ≠ A, 4-2 and C ≠ A by Axiom 4-1 1(b).

4-1 Furthermore, 4-2 B ≠ C by Theorem 4-1 3. Now, 4-1 there is a line l containing B and C by Axiom 2. This line l can not 4-1 contain A since if it did 4-1 we would have A, B, and C all on 4-2 l and l = m = n 4-1 which is impossible. Thus, we have established the above theorem. 4-1

Exercises 10-1 — 0

Prove the following 5-1 theorems. — 0

1. Theorem 5: If A is a point in plane π then there are at least two
0 — 9-1 lines in π each containing point A. — 0

2. Theorem 6: There are at 9-1 least three non-concurrent lines in plane π.

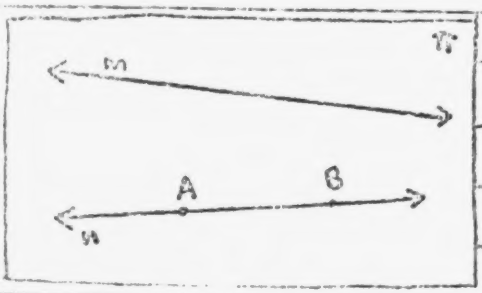
3. Theorem 7: If each of two lines in π is parallel to the same line in π
0 — 9-1 then they are parallel to each other. — 0

4. Theorem 8: If m is any line in plane π then there are at least two
0 — 9-1 points in π which are not in line m. — 0

5. Theorem 9: If A is any point in plane π then there are at least two
0 — 9-1 lines in π which do not contain A. — 0

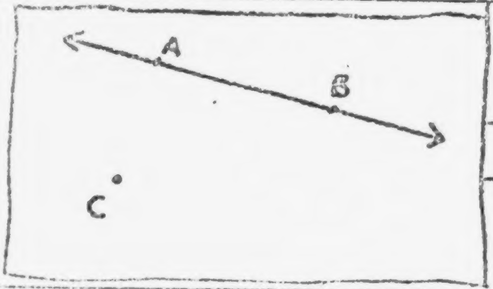
Practice Passage 8

Some Logical Consequences of the Axioms



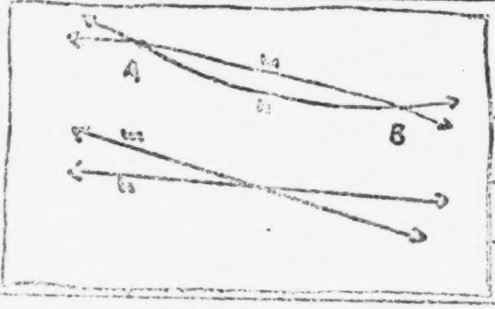
Suppose m is a line in plane π. By Axiom 1(a) there is a line n in π with $n \neq m$. By Axiom 1(b) there are distinct points A and B in line n. Now if both A and B were in m then we would have $m = n$ which is not the case. Hence, at least one of the points A or B is not in m. Thus, we have the following theorem.

Theorem 1: If m is a line in plane π then there is a point in π which is not in m.



Now, by Axiom 1(a) there exists a line m in plane π. By Axiom 1(b) there are distinct points A and B on m. By Theorem 1 there exists a point C in plane π but not in line m. Hence, we have our second theorem.

Theorem 2: There are at least three non-collinear points in a given plane.

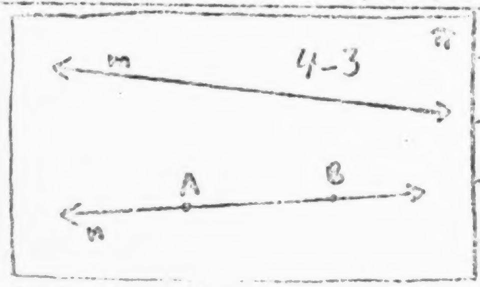


Let m and n be two distinct lines in a plane π. We know such lines exist by Axiom 1(a). Assume there are two distinct points A and B so that $A \in m \cap n$ and $B \in m \cap n$. Then m and n are two distinct lines each containing points A and B. But, this is impossible by Axiom 2. Hence, the above assumption is impossible and we have proven our third theorem.

Theorem 3: Two distinct lines can not have more than one point in common.

Key to Practice Passage 8

Some Logical Consequences of the Axioms 10-1 | 0



Suppose m is a line in plane π. 4-1

By Axiom 1(a) there is a line n in π 4-1

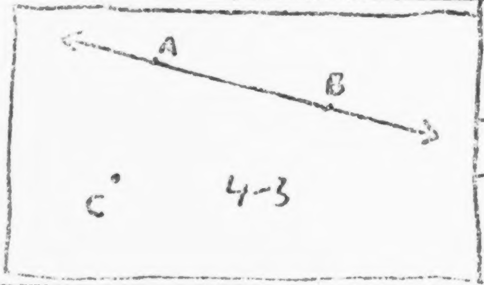
with $n \neq m$. 4-2 | By Axiom 1(b) there are distinct points A and B in line n. 4-1

Now if both A and B were in m then 4-1

we would have 4-1 | $m = n$ 4-2 | which is not the case. 4-1 | Hence, at least one of the 4-1

points A or B is not in m. 4-1 | Thus, we have the following theorem. 4-1 | 0

Theorem 1: 2-1 | If m is a line in plane π then there is a point in π which is not in m. 2-1 | 0 | 0

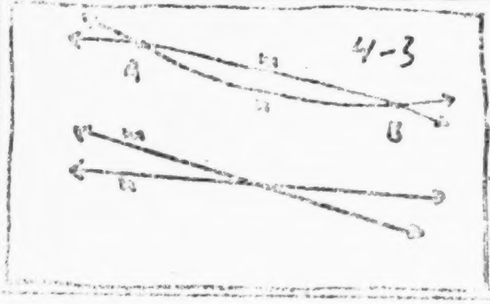


Now, by Axiom 1(a) there exists a line m in plane π. 4-1 | By Axiom 1(b) there are distinct points A and B on m. 4-1

By Theorem 1 there exists a point C in plane π but not in line m. 4-1 | Hence, 4-1

we have our second theorem. 4-1 | 0

Theorem 2: 2-1 | There are at least three non-collinear points in a given plane. 2-1



Let m and n be two distinct lines in a plane π. 4-1 | We know such lines exist by Axiom 1(a). 4-1 | Assume there are two distinct points A and B so that 4-1 | $A \in m \cap n$ 4-2

and $B \in m \cap n$. 4-2 | Then m and n are two 4-1

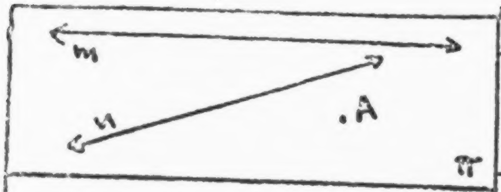
distinct lines each containing points A and B. 4-1 | But, this is impossible 4-1

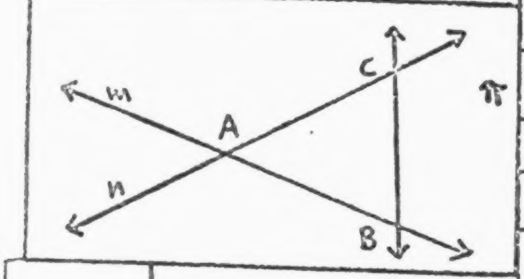
by Axiom 2. 2-1 | Hence, the above assumption is impossible and we have proven 4-1

our third theorem. 4-1 | 0

Theorem 3: Two distinct lines can not have more than one point in common.

Practice Passage 8 (cont.)





Now let A be any point in plane π. By Axiom 1(a) there are two distinct lines m and n in π. If both m and n contain A then we have points B and C with B in n, C in m, $A \neq B$, and $A \neq C$ by Axiom 1(b). Furthermore, $B \neq C$ because of Theorem 3. Therefore, there is a line l containing B and C by Axiom 2. This line l can not contain A since if it did we would have A, B, and C all on l and $l = m = n$ which is impossible. Thus we have shown that in case $A \in m \cap n$ then there is a line l in π with A not on l. Clearly, if $A \notin n \cap m$ then either m or n does not contain A. Thus we have established the following theorem.

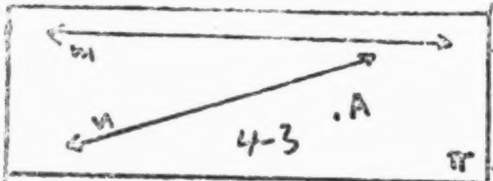
Theorem 4: If A is a point in a plane π then there is a line in π which does not contain A.

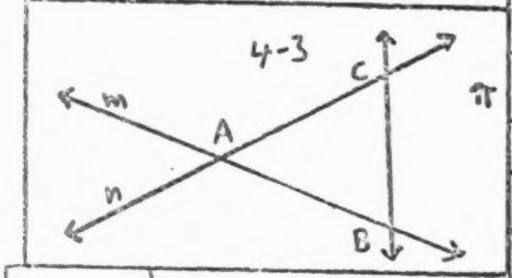
Exercises

Prove the following theorems.

1. Theorem 5: If A is a point in plane π then there are at least two lines in π each containing point A.
2. Theorem 6: There are at least three non-concurrent lines in plane π.
3. Theorem 7: If each of two lines in π is parallel to the same line in π then they are parallel to each other.
4. Theorem 8: If m is any line in plane π then there are at least two points in π which are not in line m.
5. Theorem 9: If A is any point in plane π then there are at least two lines in π which do not contain A.

Key to Practice Passage 8 (cont.)





Now let A be any point in plane π. 4-1

By Axiom 1(a) there are two distinct 4-1

lines m and n in π.

If both m and n contain A then we have points B and C 4-1

with B in n, C in m, A ≠ B, and A ≠ C 4-2

by Axiom 1(b). 4-1

Furthermore, 4-2

B ≠ C 4-2

because of Theorem 3. 4-1

Therefore, there is a line l containing B and C by 4-1

Axiom 2.

This line l can not contain A since if it did we would have 4-1

A, B, and C all 4-2 l and l = m = n

which is impossible. 4-1

Thus we have 4-1

shown that in case 4-1

A ∈ m ∩ n 4-2

then there is a line l in π with A not on l. 4-1

Clearly, if A ∉ n ∩ m 4-2

then either m or n does not contain A. 4-1

Thus we have established the following theorem. 4-1

0

Theorem 4: If A is a point in a plane π then there is a line in π

0

which does not contain A. 2-1

0

Exercises 10-1

0

Prove the following theorems. 5-1

0

1. Theorem 5: If A is a point in plane π then there are at least two

0

lines in π each containing point A. 9-1

0

2. Theorem 6: There are at least three non-concurrent lines in plane π. 9-1

3. Theorem 7: If each of two lines in π is parallel to the same line in π

0

then they are parallel to each other. 9-1

0

4. Theorem 8: If m is any line in plane π then there are at least two

0

points in π which are not in line m. 9-1

0

5. Theorem 9: If A is any point in plane π then there are at least two

0

lines in π which do not contain A. 9-1

0

Practice Passage 9

4.6. Graph of a Function. The graph of a function is the set of points each point of which has as its first coordinate an element of the domain and as its second coordinate the corresponding element of the range. According to the definition of a function, an element of the domain is a coordinate of one and only one point of the graph.

Given a graph, we may desire to know whether it could be the graph of a function. If only the graph is given, we would assume that the graph defines the domain and range. We would need to check to see whether the graph could be used as the rule for a function.

Also, given a domain and range, we may desire to know whether the graph could be used as the rule for a function. One such check is made by the use of a vertical line. If the number 1 is an element of the domain and elements of the domain are represented on the X-axis, place a vertical line through 1 on the X-axis. If this line intersects more than one point of the graph, the graph is not the rule for a function. This is true for any element of the domain.

An alternate check may be used if the number of points on the graph is finite. Write the coordinates of the points on the graph. If any first coordinate is used for two different points, the graph is not the rule for a function.

Examples

1. Domain: {1, 2, 3, 4}
   Range: {1, 2, 3}
   Could the graph in Fig. 4.4 be the rule for a function using this domain and range?

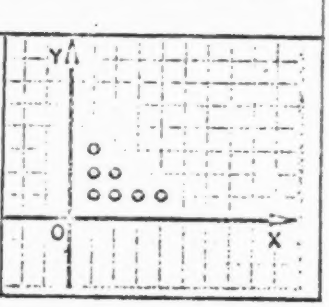


Fig. 4.4

(Reprinted by permission.
Contemporary Algebra, Second Course,
Mayor and Wilcox, Prentice Hall, 1965.)

Key to Practice Passage 9

4.6. Graph of a Function. The graph of a function is the set of points each point of which has as its first coordinate an element of the domain and as its second coordinate the corresponding element of the range. According to the definition of a function, an element of the domain is a coordinate of one and only one point of the graph.

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0

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An alternate check may be used if the number of points on the graph is finite. Write the coordinates of the points on the graph. If any first coordinate is used for two different points, the graph is not the rule for a function.

0

Examples

0

1. Domain: {1, 2, 3, 4}
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   Could the graph in Fig. 4.4 be the rule for a function using this domain and range?

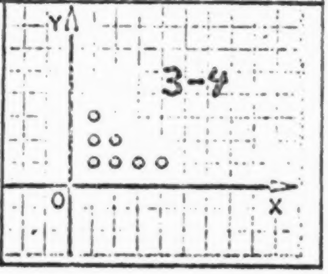


Fig. 4.4

The answer is no. A vertical line through 1 on the X-axis passes through more than one point on the graph. Since there are a finite number of points on the graph, we could write the coordinates of these points: (1,1), (1,2) (1,3), (2,1), (2,2), (3,1), (4,1). The numbers 1 and 2 are used as first coordinates of more than one point.

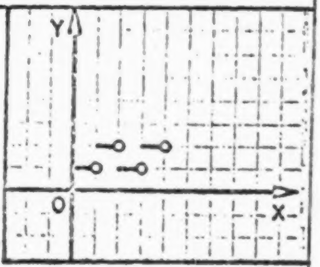
2. **Graph the following function.**

Domain: $\{x | x \in R, 0 \leq x < 4\}$

Range: $\{y | y \in R\}$

Rule:

$$\begin{cases} 0 \leq x < 1, y = 1 \\ 1 \leq x < 2, y = 2 \\ 2 \leq x < 3, y = 1 \\ 3 \leq x < 4, y = 2 \end{cases}$$



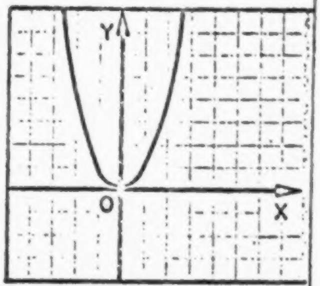
3. **Graph the following function.**

Domain: R

Range: R

Rule: $y = x^2$

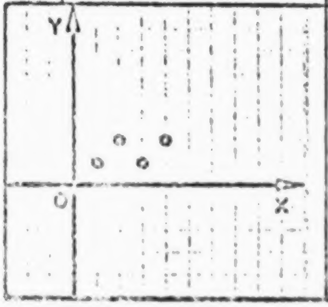
The complete graph cannot be shown.



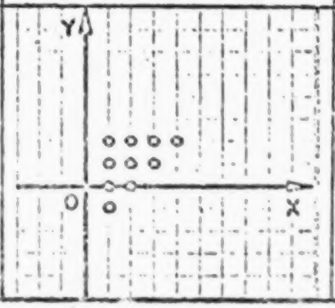
Exercises

Which of the graphs in Exercises 1 through 12 could be the rule for a function? In Exercises 1 through 6 the domain is $\{1, 2, 3, 4\}$ and the range is $\{-1, 0, 1, 2\}$.

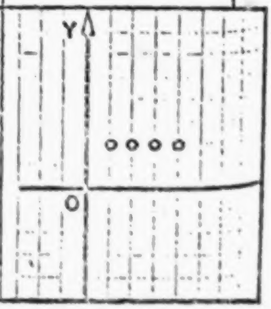
1.



2.



3.



Key to Practice Passage 9 (cont.)

The answer is no. A vertical line through 1 on the X-axis passes through more than one point on the graph. Since there are a finite number of points on the graph, we could write the coordinates of these points: (1,1), (1,2) (1,3), (2,[illegible]), (2,2), (3,1), (4,1). The numbers 1 and 2 are used as first coordinates of more than one point.

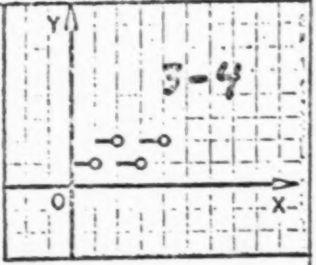
0

2. Graph the following function.

Domain: $\{x \mid x \in R, 0 \leq x < 4\}$
Range: $\{y \mid y \in R\}$
Rule:
$0 \leq x < 1, y = 1$
$1 \leq x < 2, y = 2$
$2 \leq x < 3, y = 1$
$3 \leq x < 4, y = 2$

3-2

0



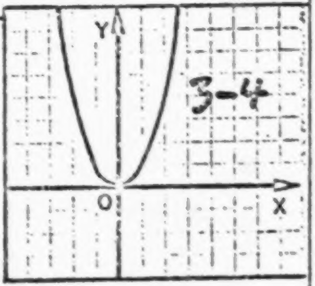
3. Graph the following function.

Domain: R
Range: R
Rule: $y = x^2$

3-2

The complete graph cannot be shown.

0



Exercises

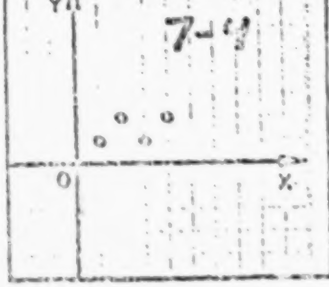
0

Which of the graphs in Exercises 1 through 12 could be the rule for a function? In Exercises 1 through 6 the domain is $\{1, 2, 3, 4\}$ and the range is $\{-1, 0, 1, 2\}$.

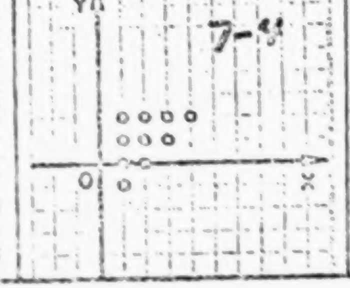
3-2

0

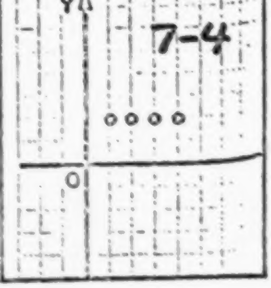
1. 0



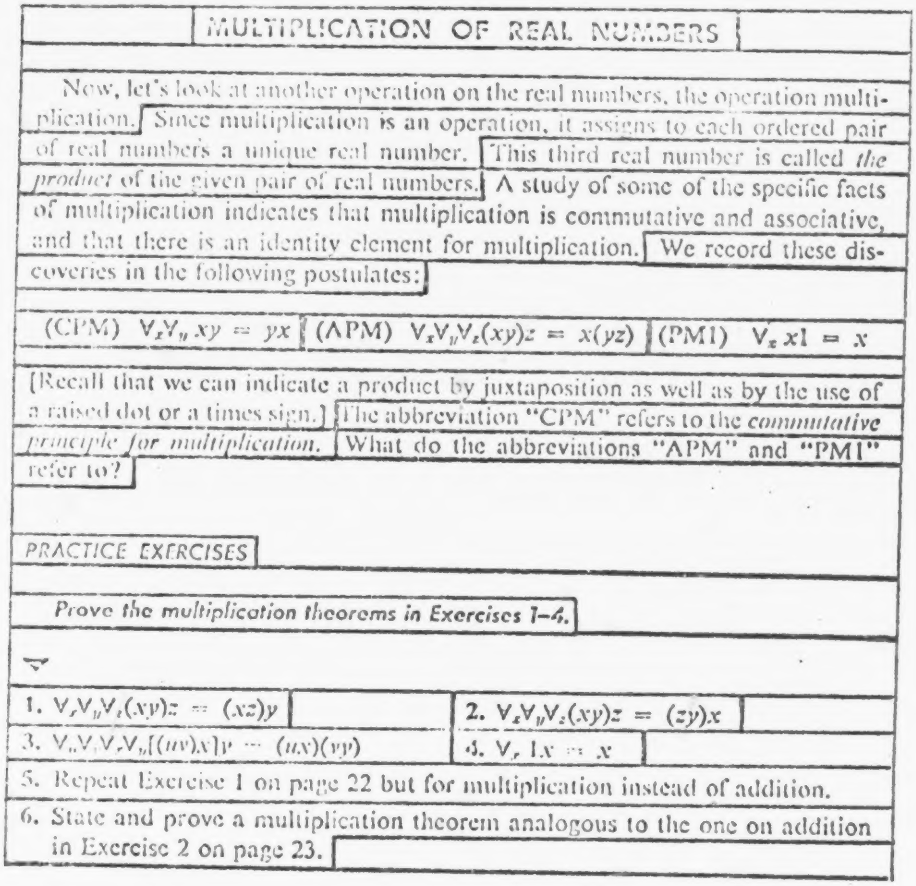
2. 0



0 3.



Practice Passage 10



MULTIPLICATION OF REAL NUMBERS

Now, let's look at another operation on the real numbers, the operation multiplication. Since multiplication is an operation, it assigns to each ordered pair of real numbers a unique real number. This third real number is called *the product* of the given pair of real numbers. A study of some of the specific facts of multiplication indicates that multiplication is commutative and associative, and that there is an identity element for multiplication. We record these discoveries in the following postulates:

(CPM) $\forall_x\forall_y\, xy = yx$ (APM) $\forall_x\forall_y\forall_z(xy)z = x(yz)$ (PMI) $\forall_x\, x1 = x$

[Recall that we can indicate a product by juxtaposition as well as by the use of a raised dot or a times sign.] The abbreviation "CPM" refers to the *commutative principle for multiplication.* What do the abbreviations "APM" and "PMI" refer to?

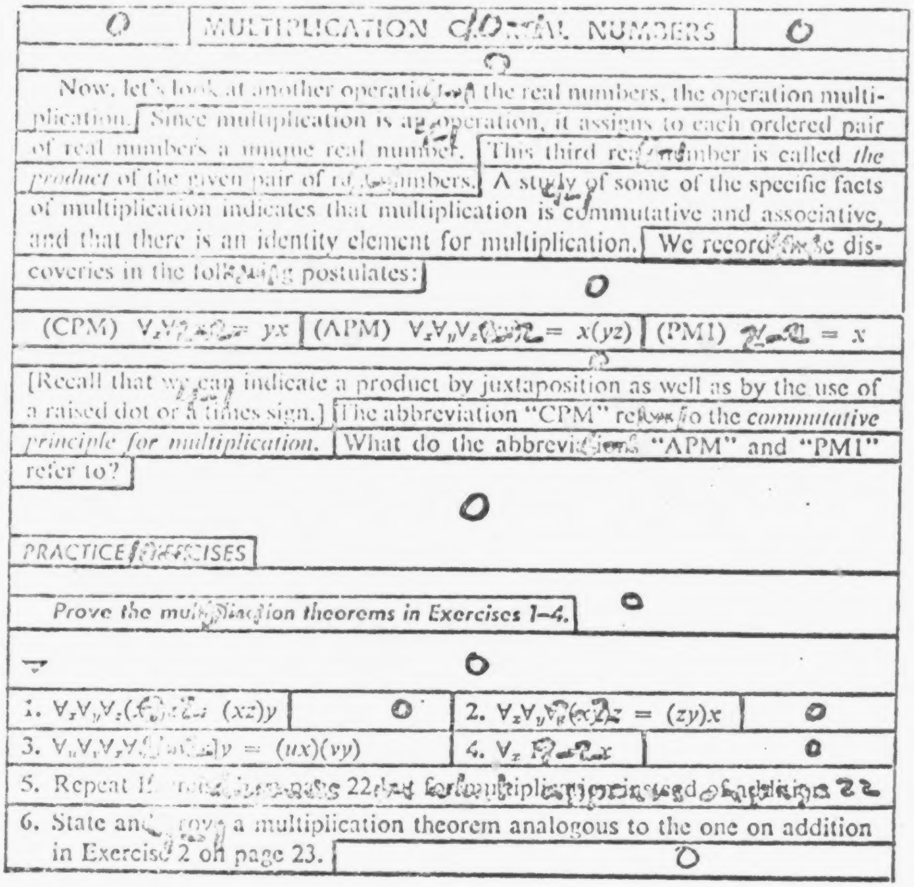
PRACTICE EXERCISES

Prove the multiplication theorems in Exercises 1–4.

1. $\forall_x\forall_y\forall_z(xy)z = (xz)y$
2. $\forall_x\forall_y\forall_z(xy)z = (zy)x$
3. $\forall_u\forall_v\forall_x\forall_y[(uv)x]y = (ux)(vy)$
4. $\forall_x\, 1x = x$
5. Repeat Exercise 1 on page 22 but for multiplication instead of addition.
6. State and prove a multiplication theorem analogous to the one on addition in Exercise 2 on page 23.

(Reprinted by permission.
Algebra with Trigonometry,
Fehr et al., Heath, 1963.)

Key to Practice Passage 10



O MULTIPLICATION [illegible] NUMBERS O

O

Now, let's look at another operation [illegible] the real numbers, the operation multiplication. Since multiplication is an operation, it assigns to each ordered pair of real numbers a unique real number. This third real number is called *the product* of the given pair of real numbers. A study of some of the specific facts of multiplication indicates that multiplication is commutative and associative, and that there is an identity element for multiplication. We record these discoveries in the following postulates:

O

(CPM) $\forall_x\forall_y$ [illegible] $= yx$ (APM) $\forall_x\forall_y\forall_z$ [illegible] $= x(yz)$ (PMI) [illegible] $= x$

O

[Recall that we can indicate a product by juxtaposition as well as by the use of a raised dot or a times sign.] The abbreviation "CPM" refers to the *commutative principle for multiplication.* What do the abbreviations "APM" and "PMI" refer to?

O

PRACTICE EXERCISES

Prove the multiplication theorems in Exercises 1–4.

O

O

1. $\forall_x\forall_y\forall_z($[illegible]$) = (xz)y$ O
2. $\forall_x\forall_y\forall_z(x$[illegible]$)z = (zy)x$ O
3. $\forall_u\forall_v\forall_x\forall_y$[illegible]$y = (ux)(vy)$
4. $\forall_x$ [illegible] x O
5. Repeat E[illegible] page 22 [illegible] multiplication instead of addition. 22
6. State and prove a multiplication theorem analogous to the one on addition in Exercise 2 on page 23. O

THE PRINCIPLE FOR ADDING 0

The postulates and theorems on addition we have collected so far deal with those properties of addition that concern the order in which addition is performed. There is another property of addition which is quite different.

Recall that addition is an operation which assigns a unique real number to each ordered pair of real numbers. Now, there are certain ordered pairs of real numbers to which addition assigns the first number of the ordered pair. For example, addition assigns 8 to $(8, 0)$, $^-9$ to $(^-9, 0)$, and π to $(\pi, 0)$. In fact, no matter what real number you pick, you can always find a second real number to add to it so that the sum is identical with the first real number chosen — that is,

$$\forall_x \exists_y\; x + y = x.$$

Moreover, there is a second real number which will work for all cases. In other words,

$$\exists_y \forall_x\; x + y = x.$$

This last generalization is a short way of saying that the operation addition has an *identity element.* And, as we know, 0 is such an identity element for addition. We express this property of addition in the postulate:

$$\text{(PA0)} \quad \forall_x\; x + 0 = x$$

which is called *the principle for adding 0.*

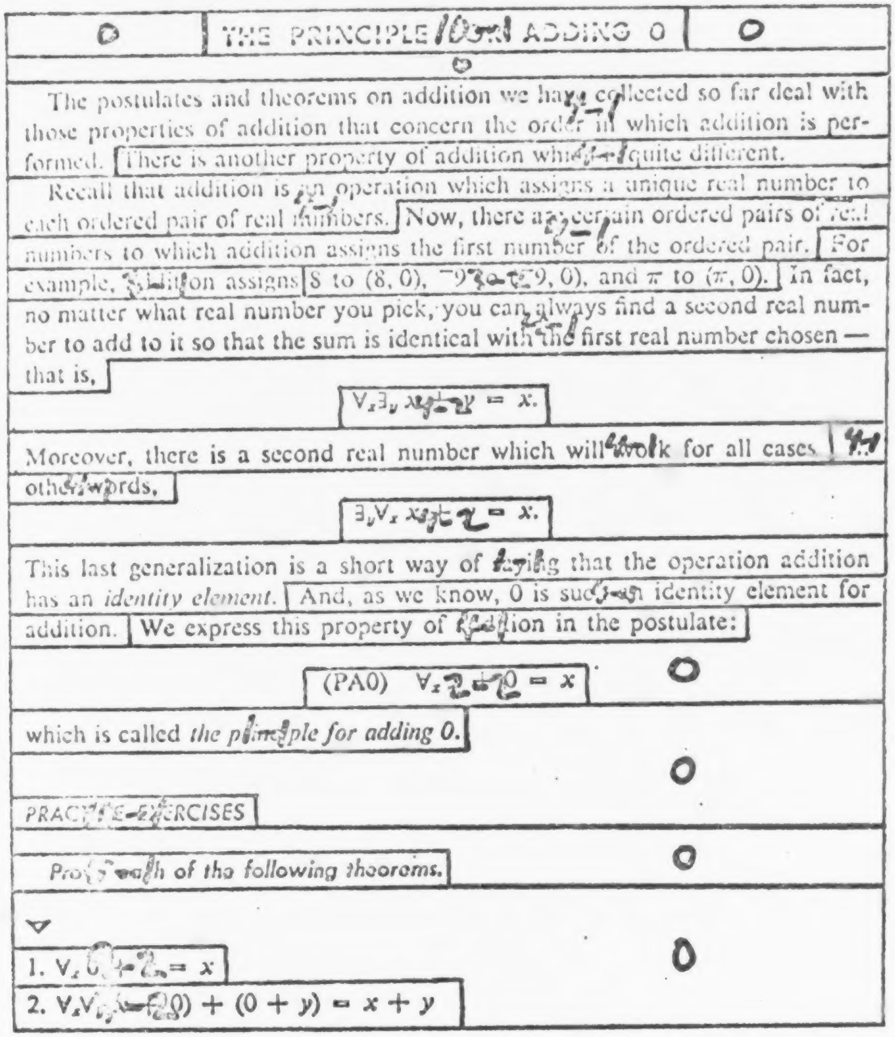
PRACTICE EXERCISES

Prove each of the following theorems.

▽

1. $\forall_x\; 0 + x = x$
2. $\forall_x \forall_y\; (x + 0) + (0 + y) = x + y$

Key to Practice Passage 10 (cont.)



0 THE PRINCIPLE FOR ADDING 0 0

0

The postulates and theorems on addition we have collected so far deal with those properties of addition that concern the order in which addition is performed. There is another property of addition which is quite different.

Recall that addition is an operation which assigns a unique real number to each ordered pair of real numbers. Now, there are certain ordered pairs of real numbers to which addition assigns the first number of the ordered pair. For example, addition assigns 8 to (8, 0), ⁻9 to (⁻9, 0), and π to $(\pi, 0)$. In fact, no matter what real number you pick, you can always find a second real number to add to it so that the sum is identical with the first real number chosen — that is,

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$$(\text{PA0}) \quad \forall_x\, x + 0 = x$$ 0

which is called *the principle for adding 0.*

0

PRACTICE EXERCISES

Prove each of the following theorems. 0

0

1. $\forall_x\, 0 + x = x$
2. $\forall_x \forall_y\, (x + 0) + (0 + y) = x + y$

Practice Passage 11

Cartesian Products and Relations

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Forming the set of all ordered pairs whose first component is from A and whose second component is from B we have $\{(1,a),(2,a),(3,a),(1,b),(2,b),(3,b)\}$. This set is denoted $\{1, 2, 3\} \times \{a,b\}$ or $A \times B$. $A \times B$ is read " A cross B" and is called the cartesian product of A and B. Notice that in this example an ordered pair $(x,y) \in \{1, 2, 3\} \times \{a, b\}$ if and only if $x \in \{1, 2, 3\}$ and $y \in \{a, b\}$.

Now letting $M = \{+, \triangle\}$ and $N = \{\triangle, \star, \square\}$ we have, $M \times N = \{(+,\triangle), (+,\star), (+,\square), (\triangle,\triangle), (\triangle,\star), (\triangle,\square)$. Notice that in this example $M \cap N = \{\triangle\} \neq \phi$. But, if the example above $A \times B = \phi$.

* * *

1. Give the roster name for each of the following cartesian products.

a. $\{2, 3, 7\} \times \{0, 2, 3, 5, 6\}$

b. $Z_3 \times Z_5$

c. $\{x \mid x \in W \text{ and } x < 4\} \times \{z \mid z \in W \text{ and } z + 5 \leq 11\}$

d. $A \times A$ where $A = \{x \mid x \in W, x \text{ is even, and } x < 9\}$

* * *

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$.

Let $R = \{(1,a), (2,b), (3,a)\}$.

Then $R \subset A \times B$. We say the set R is a relation from A to B.

Suppose $S = \{(1,b), (3,b)\}$. Since $S \subset A \times B$, then S is also a relation from A to B.

* * *

2. Make up a relation from the first set to the second set in each part of exercise 1.

3. Make up a relation from the second set to the first set in each part of exercise 1.

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Key to Practice Passage 11

Cartesian Products and Relations 0

3-2 Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Forming the set 3-1 of all ordered

pairs whose first component is from A and whose second component is from

B we have $\{(1,a),(2,a),(3,a),(1,b),(2,b),(3,b)\}$. This set 3-1 is denoted

$\{1, 2, 3\} \times \{a, b\}$ or $A \times B$. $A \times B$ is read " A cross B" and is called

the cartesian product of A and B. Notice that in this example an ordered

pair $(x,y) \in \{1, 2, 3\} \times \{a, b\}$ if and only if $x \in \{1, 2, 3\}$ and $y \in \{a, b\}$

3-1 Now letting $M = \{+, \Delta\}$ and $N = \{\Delta, \#, \square\}$ we have, 3-1

$M \times N = \{(+,\Delta), (+,\#), (+,\square), (\Delta,\Delta), (\Delta,\#), (\Delta,\square)\}$. Notice that in this

example $M \cap N = \{\Delta\} \neq \phi$. But, if the example above $A \times B = \phi$. 3-2

* * * 0

1. Give the roster name for each of the following cartesian products.

3-2 a. $\{2, 3, 7\} \times \{0, 2, 3, 5, 6\}$ 0

3-2 b. $Z_3 \times Z_5$ 0

3-2 c. $\{x \mid x \in W \text{ and } x < 4\} \times \{z \mid z \in W \text{ and } z + 5 \leq 11\}$ 0

3-2 d. $A \times A$ where $A = \{x \mid x \in W, x \text{ is even, and } x < 9\}$ 0

* * * 0

3-2 Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$.
Let $R = \{(1,a), (2,b), (3,a)\}$. 0

3-2 Then $R \subset A \times B$. We say that set R is a relation from A to B.

Suppose $S = \{(1,b), (3,b)\}$. Since $S \subset A \times B$, then S is also a relation

from A to B. 0

* * * 0

2. Make up a relation from the first set to the second set in each part of exercise 1. 0

3. Make up a relation from the second set to the first set in each part of exercise 1. 0

Practice Passage 11 (cont.)

Let $A = \{x \mid x$ is a male human being$\}$

Let $B = \{y \mid y$ is a female human being$\}$

Let $R = \{(a, b) \mid b$ is the daughter of a and a is the father of b$\}$.

Then $R \subset A \times B$ and hence R is a relation from A to B. We might call this relation the father-daughter relation.

Let $U = \{x \mid x$ is a human being$\}$

Let $S = \{(a,b) \mid a$ is the brother of b$\}$

Then $S \subset U \times U$ and S is a relation from U to U. Notice that the set U is used as both the first and second set in the cartesian product in this example. In such a case we say S is a relation on U.

* * *

4. List two elements of the father-daughter relation.

5. List two elements of the brother relationship.

6. Which of the following sets are relations? Give the roster names of those which are relations.

   a. $\{2x + 3y \mid x \in W, y \in W, x < 5,$ and $y < 3\}$

   b. $Z_3 \times Z_4$

   c. $\{(x + 5, 3x) \mid x \in W$ and $x < 3\}$

   d. $\{1, 2, 6\} \times \{-1, -4, 3, 2\}$

   e. ϕ

* * *

Let W be the set of whole numbers.

Let $L = \{(x,y) \mid \exists\, c \in N$ such that $x + c = y\}$

Since $L \subset W \times W$ then L is a relation on W. L is such an important relation on W that we give it a special name. L is called the "less than" relation on W. We usually denote L by $<$ and write $2 < 3$ or $2 \; L \; 3$ to indicate that $(2,3) \in L$. Both of the statements $5 \; L \; 9$ and $(5,9) \in L$ mean the same thing.

Key to Practice Passage 11 (cont.)

3-2 Let $A = \{x \mid x \text{ is a male human being}\}$ 3-2 Let $B = \{y \mid y \text{ is a female human being}\}$	0
3-2 Let $R = \{(a,b) \mid b \text{ is the daughter of } a \text{ and } a \text{ is the father of } b\}$.	
3-2 Then $R \subset A \times B$ and hence R is a relation from A to B. We might call	
this relation the father-daughter relation.	0
3-2 Let $U = \{x \mid x \text{ is a human being}\}$ 3-1 3-2 Let $S = \{(a,b) \mid a \text{ is the brother of } b\}$ 3-1	0
3-2 Then $S \subset U \times U$ and S is a relation from U to U. Notice that the	
set U is used as both the first and second set in the cartesian product	
in this example. In such a case we say S is a relation on U.	0
* * *	0
4. List two elements of the father-daughter relation.	0
5. List two elements of the brother relationship.	0
6. Which of the following sets are relations? Give the proper names	
of those which are relations.	0
7-2 a. $\{2x + 3y \mid x \in W, y \in W, x < 5, \text{ and } y < 3\}$	0
7-2 b. $Z_3 \times Z_4$	0
7-2 c. $\{(x + 5, 3x) \mid x \in W \text{ and } x < 3\}$	0
7-2 d. $\{1, 2, 6\} \times \{-1, -4, 3, 2\}$	0
7-2 e. ϕ	0
* * *	0
3-1 Let W be the set of whole numbers.	0
3-2 Let $L = \{(x,y) \mid \exists\, c \in N \text{ such that } x + c = y\}$	0
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that $(2,3) \in L$. Both of the statements $5 \text{ L } 9$ and $(5,9) \in L$ mean the same	
thing.	0

Practice Passage 11 (cont.)

* * *

7. a. Define the relation $>$ on W in terms of ordered pairs.

b. List five ordered pairs in the relation $>$ on W.

c. Rewrite $(a,b) \in >$ as $a > b$ for each of your answers to b.

8. Rewrite your answers to exercises 4 and 5 in the form $x \; R \; y$ where R represents the relation given in each problem.

* * *

Key to Practice Passage 11 (cont.)

* * * 0

7. 6-0 Define the relation > on W in terms of ordered pairs.

6-1. List five ordered pairs in the relation > on W.

6-2 Rewrite (a,b)ε> as a > b for each of your answers to b.

8. Rewrite 7-0 your answers to exercises 4 and 5 in the form x R y 7-2 where

7-1 R represents the relation given in each problem.

* * * 0

Practice Passage 12

Cartesian Products and Relations

Consider arbitrary sets A and B and the set of all ordered pairs whose first component is an element of A and whose second component is an element of B. Such sets of ordered pairs are important in many branches of mathematics. We now make the following formal definition.

Definition 1: For all sets A and B,

$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$.

$A \times B$ is read " A cross B " and is called the cartesian product of A and B.

An immediate consequence of this definition is the following theorem.

Theorem 1: $(a,b) \in A \times B \Leftrightarrow a \in A \text{ and } b \in B$.

Proof: $(\Rightarrow)$ Let $(a,b) \in A \times B$. Then $(a,b) \in \{(x,y) \mid x \in A \text{ and } y \in B\}$.

So, $a \in A$ and $b \in B$.

$(\Leftarrow)$ Let $a \in A$ and $b \in B$. Then $(a,b) \in \{(x,y) \mid x \in A \text{ and } y \in B\} = A \times B$.

Definition 2: Let A and B be arbitary sets.

Let $R \subset A \times B$.

Then R is called a relation from A to B.

This definition tells us that any subset of a cartesian product of two sets is a relation from the first set to the second set. When $R \subset A \times A$ we have a relation from A to A. In this case we say R is a relation on A. The idea of a relation is an important mathematical concept. Many relations can be found and are used in everyday life. Among these are all the biological relationships between people.

Let R be a relation. Since $R \subseteq A \times B$ for some sets A and B we know the elements of R must be ordered pairs. Thus we can write $(a,b) \in R$ to show the ordered pair (a,b) is in the relation R. Many times we write a R b to indicate that $(a,b) \in R$. Both of these statements mean exactly the same thing.

Key to Practice Passage 12

Cartesian Products and Relations | 0

Consider arbitrary sets A and B and the set of all ordered pairs whose

4-1

first component is an element of A and whose second component is an element

of B. | Such sets of ordered pairs are important in many branches of mathe-

matics. | We now make the following formal definition. | 0

Definition 1: For all sets A and B, | 0

0 | $A \times B = \{(x,y) \mid x \in A$ and $y \in B\}$.

$A \times B$ | is read " A cross B " and is called the cartesian

product of A and B.

An immediate consequence of this definition is the following theorem.

Theorem 1: $(a,b) \in A \times B \Leftrightarrow a \in A$ and $b \in B$. | 0

Proof: $(\Rightarrow)$ Let $(a,b) \in A \times B$. | Then $(a,b) \in \{(x,y) \mid x \in A$ and $y \in B\}$.

0 | So, $a \in A$ and $b \in B$. | 0

0 | $(\Leftarrow)$ Let $a \in A$ and $b \in B$. | Then $(a,b) \in \{(x,y) \mid x \in A$ and $y \in B\} = A \times B$.

Definition 2: Let A and B be arbitrary sets. | 0

0 | Let $R \subset A \times B$.

Then R is called a relation from A to B.

This definition tells us that any subset of a cartesian product of two

4-1

sets is a relation from the first set to the second set. | When $R \subset A \times A$ 4-2

we have a relation from A to A. | In this case we say R is a relation on A.

The idea of a relation is an important mathematical concept. | Many relations

can be found and are used in everyday life. | Among these are all the

biological relationships between people. | 0

Let R be a relation. | Since $R \subset A \times B$ | for some sets A and B we know

the elements of R must be ordered pairs. | Thus we can write | $(a,b) \in R$ 4-2

to show the ordered pair (a,b) is in the relation R. | Many times we write

a R b | to indicate that | $(a,b) \in R$. | Both of these statements mean exactly

the same thing. | 0

Practice Passage 12 (cont.)

Exercises

1. Give the roster name for each of the following cartesian products.

a. $\{2, 3, 7\} \times \{0, 2, 3, 5, 6\}$

b. $Z_3 \times Z_5$

c. $\{x \mid x \in W \text{ and } x < 4\} \times \{z \mid z \in W \text{ and } z + 5 \leq 11\}$

d. $A \times A$ where $A = \{x \mid x \in W, x \text{ is even, and } x < 9\}$

2. Make up a relation from the first to the second set in each part of exercise 1.

3. Make up a relation from the second set to the first set in each part of exercise 1.

4. List two elements of the father-daughter relation.

5. List two elements of the brother relation.

6. Which of the following sets are relations? Give the roster names of those which are relations.

a. $\{2x + 3y \mid x \in W, y \in W, x < 5, \text{ and } y \leq 3\}$

b. $Z_3 \times Z_5$

c. $\{(x + 5, 3x) \mid x \in W \text{ and } x \leq 3\}$

d. $\{1, 2, 3\} \times \{-1, -4, 3, 2\}$

e. ϕ

7. a. Define the relation "less than" on W in terms of ordered pairs.

b. List five ordered pairs in the relation "less than" on W.

c. Rewrite $(a,b) \in$ "less than" as $a < b$ for each of your answers to b.

8. Repeat exercise 7 for the relation "greater than" on W.

9. Rewrite your answers to exercises 4 and 5 in the form $x \, R \, y$ where R represents the relation given in each problem.

Key to Practice Passage 12 (cont.)

ex(2-1)s	0
1. Give the roster 5-1 for each of the following cartesian products.	
7-2a. $\{2, 3, 7\} \times \{0, 2, 3, 5, 6\}$	0
7-2b. $Z_3 \times Z_5$	0
7-2c. $\{x \mid x \in W \text{ and } x < 4\} \times \{z \mid z \in W \text{ and } z + 5 \leq 11\}$	0
7-2d. $A \times A$ where $A = \{x \mid x \in W, x \text{ is even, and } x < 9\}$	0
2. Make up a relation from the first to the second set in each part	
7-1 of exercise 1.	0
3. Make up a relation from the second set to the first set in each part	
7-1 of exercise 1.	0
7-1 List two elements of the father-daughter relation.	0
57-1 List two elements of the brother relation.	0
6. 5-1h of the following sets are relations?	Give the 5-1r names
5-1hose which are relations.	0
7-2a. $\{(x, y) \mid x \in W, y \in W, x < 5, \text{ and } y \leq 3\}$	0
7-2b. $Z_3 \times Z_5$	0
7-2c. $\{(x + 5, 3x) \mid x \in W \text{ and } x \leq 3\}$	0
7-2d. $\{1, 2, 3\} \times \{-1, -2, 3, 2\}$	0
7-2e. ϕ	0
7. 6-1efine the relation "less than" on W in terms of ordered pairs.	
6-1st five ordered pairs in the relation "less than" on W.	
6c-1Rewrite (a,b) $\in$ "le6-2han" as $a < b$	for each o6-1r answers to b.
8. 6Repeat exercise 7 for the relation "greater than" on W.	
9. 7Re1rite your answers to exercises 4 and 5 in the form $x R y$ 7-2	where 7-1
7R-1represents the relation given in each problem.	0

APPENDIX B

CRITERION PASSAGE BOOKLET

Instructions

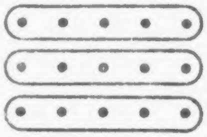
1. Use the CMMT technique to rate each of the following passages. Be sure to record a rating of the form m-n in each non-blank unit. Record 0 in each completely blank unit.
2. Rate the passages in the order presented. However, you may go back and change classifications in rated passages if you change your mind.
3. Refer to the Rater Training Booklet whenever you wish while rating the experimental passages.
4. Return this booklet to me in the envelope provided when you have completed the rating of all passages.

Focus Passage FP1

Division and sets

Here is a way to think about division.

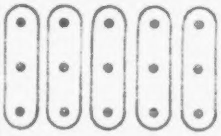
Ⓐ Think:
There are 3 sets of 5 in a set of 15.



Write: $15 \div 5 = 3$

5 is the divisor.
3 is the quotient.

Ⓑ Think:
There are 5 sets of 3 in a set of 15.



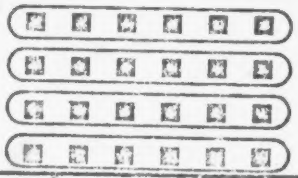
Write: $15 \div 3 = 5$

3 is the divisor.
5 is the quotient.

EXERCISES

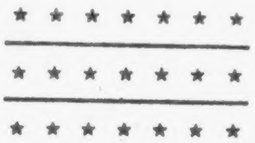
1. Draw a set of 18 dots on your paper. Ring as many sets of 6 as you can.
   [A] How many did you find? [B] Solve the equation, $18 \div 6 = \boxed{n}$
2. Draw a set of 24 dots on your paper. Ring as many sets of 4 as you can. Solve the equation, $24 \div 4 = \boxed{n}$.
3. Draw a set of 32 dots on your paper. Ring as many sets of 8 as you can. Solve the equation, $32 \div 8 = \boxed{n}$.
4. Study the sets. Then solve the equation.

[A]



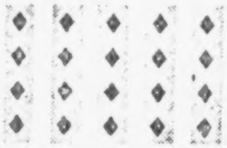
$24 \div 6 = \boxed{n}$

[B]



$21 \div 7 = \boxed{n}$

[C]



$20 \div 4 = \boxed{n}$

5. Give the missing numbers.
   [A] There are $\boxed{n}$ sixes in 36.
   $36 \div 6 = \boxed{n}$
   [B] There are $\boxed{n}$ sevens in 28.
   $28 \div 7 = \boxed{n}$

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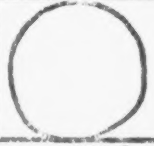
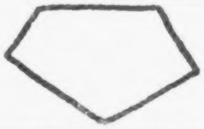
Contrived Passage CP2b

Circumference

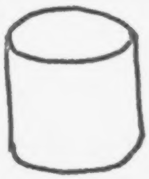
A circle like a ploygon is a plane closed figure.



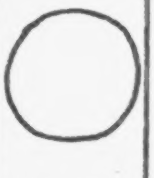


The distance around a circle is called its circumference. Circumference is to a circle as perimeter is to a polygon. You could use a ruler to find the length of each side of a pòlygon and then add these lengths to find the perimeter of the polygon. You could use a tape measure to find the circumferences and diameter lengths of the objects below.

wheel



can



plate

The table below gives the circumference, diameter length, and quotient of circumference divided by diameter length of each of these objects.

	Circumference	Diameter Length	$\frac{\text{Circumference}}{\text{Diameter Length}}$
Wheel	88 inches	28 inches	3 1/7
Can	38 inches	12 inches	3 1/6
Plate	$31\frac{3}{4}$ inches	10 inches	3 7/40

Notice that while the circumferences differ and the diameter lengths differ, the quotients of circumference divided by diameter length do not differ very much. it is only because of inaccuracy of measurement that the numbers in the last column differ at all!

It is true that when the circumference of any circle is divided by the length of a diameter of that circle, the quotient is always the same number. This number is named by the Greek letter π (pi). The number π is about $3\frac{1}{7}$. Notice how close to $3\frac{1}{7}$ each of the numbers in the final column of the above table is.

If C stands for the circumference of a circle and d stands for the diameter then

a. $\frac{C}{d} = \pi$

b. $C = d \cdot \pi$

and c. $d = \frac{C}{\pi}$

You can use formula (b) with $3\frac{1}{7}$ for π to find the circumference of a circle when you know the length of a diameter. You can use formula (c) to find the length of a diameter of a circle when you know the circumference.

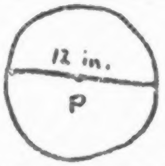
Exercises

1. Complete the table.

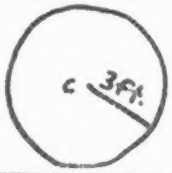
Circumference	Diameter	Radius
	5 inches	
		4 feet
33 inches		
	d	
		r
C		

Find the circumference of each circle in 2 through 9.

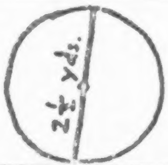
2.



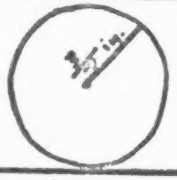
3.



4.



5.



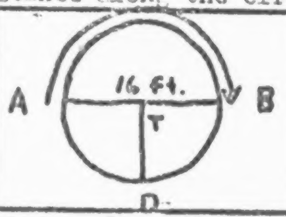
6. d = 17 yards

7. r = ½ inch

8. d = 7/11 inch

9. r = 3/4 mile

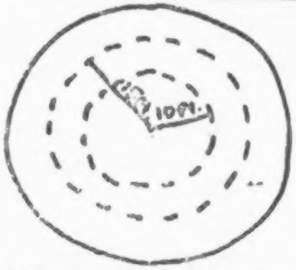
10. a. What is the distance along the circle from A to B?



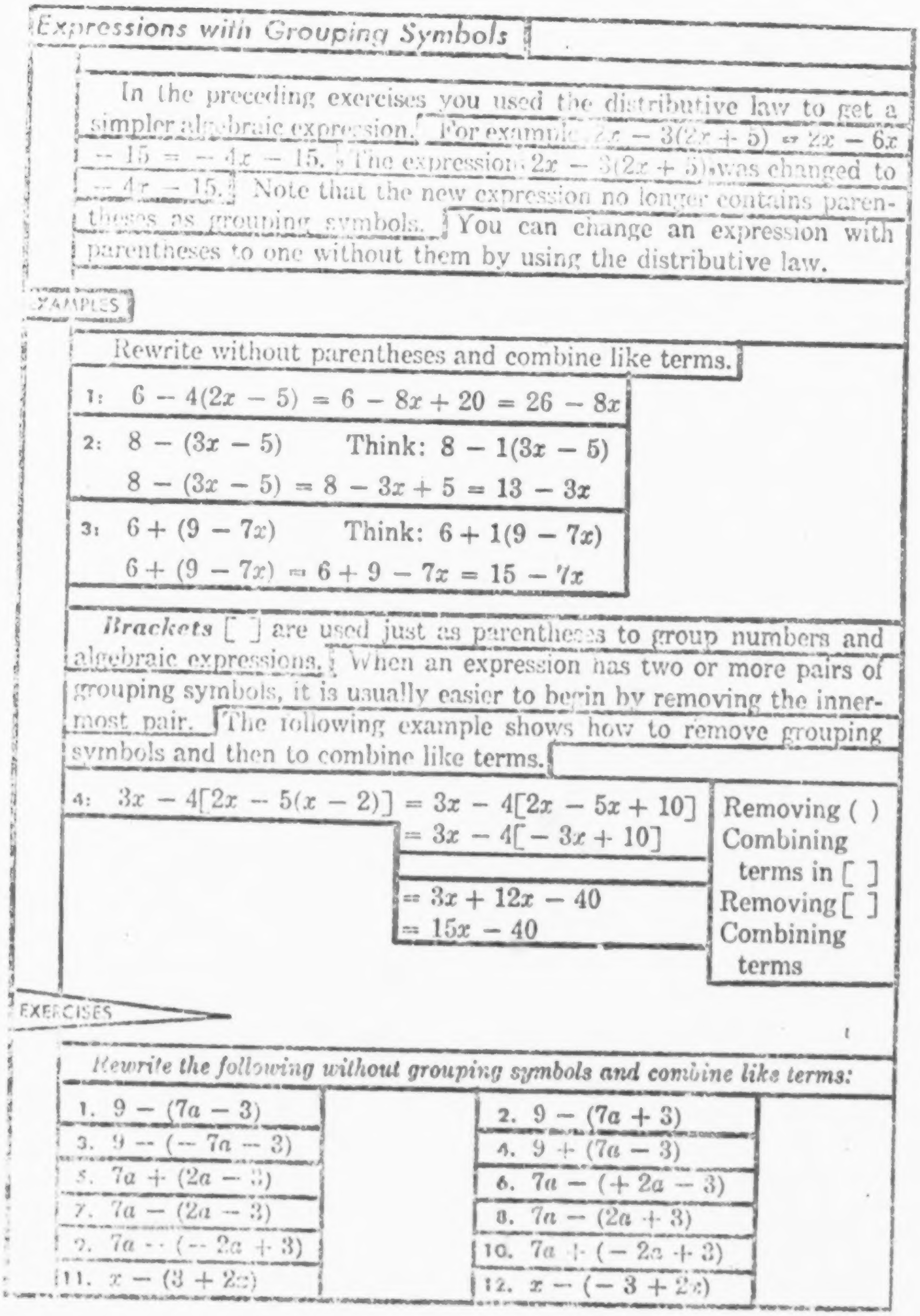
b. What is the distance along the circle from A to D if you travel in the direction of the arrow?

c. What is the distance from A to D if you travel in the opposite direction?

11. A merry-go-round makes 30 complete turns. How many feet will Ronald and Martha each travel if Martha's horse is on the inner ring and Ronald's is on the outer ring?



Focus Passage FP8



Expressions with Grouping Symbols

In the preceding exercises you used the distributive law to get a simpler algebraic expression. For example, $2x - 3(2x + 5) = 2x - 6x - 15 = -4x - 15$. The expression $2x - 3(2x + 5)$ was changed to $-4x - 15$. Note that the new expression no longer contains parentheses as grouping symbols. You can change an expression with parentheses to one without them by using the distributive law.

XAMPLES

Rewrite without parentheses and combine like terms.

1: $6 - 4(2x - 5) = 6 - 8x + 20 = 26 - 8x$

2: $8 - (3x - 5)$ Think: $8 - 1(3x - 5)$

$8 - (3x - 5) = 8 - 3x + 5 = 13 - 3x$

3: $6 + (9 - 7x)$ Think: $6 + 1(9 - 7x)$

$6 + (9 - 7x) = 6 + 9 - 7x = 15 - 7x$

Brackets [ ] are used just as parentheses to group numbers and algebraic expressions. When an expression has two or more pairs of grouping symbols, it is usually easier to begin by removing the innermost pair. The following example shows how to remove grouping symbols and then to combine like terms.

4: $3x - 4[2x - 5(x - 2)] = 3x - 4[2x - 5x + 10]$ Removing ( )

$= 3x - 4[-3x + 10]$ Combining terms in [ ]

$= 3x + 12x - 40$ Removing [ ]

$= 15x - 40$ Combining terms

EXERCISES

Rewrite the following without grouping symbols and combine like terms:

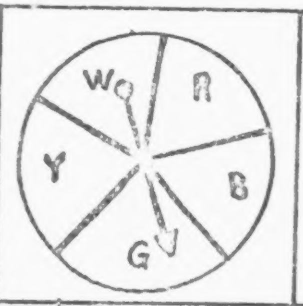
1. $9 - (7a - 3)$
2. $9 - (7a + 3)$
3. $9 - (-7a - 3)$
4. $9 + (7a - 3)$
5. $7a + (2a - 3)$
6. $7a - (+2a - 3)$
7. $7a - (2a - 3)$
8. $7a - (2a + 3)$
9. $7a - (-2a + 3)$
10. $7a + (-2a + 3)$
11. $x - (3 + 2x)$
12. $x - (-3 + 2x)$

(Reprinted by permission. Contemporary Algebra, Book One, Smith, et al., Harcourt, Brace and World, Inc., 1962.)

Contrived Passage CP6a

Probability of A or B

Spin the pointer on the dial below.



R = red

B = blue

G = green

Y = yellow

W = white

P(B) = 1/5

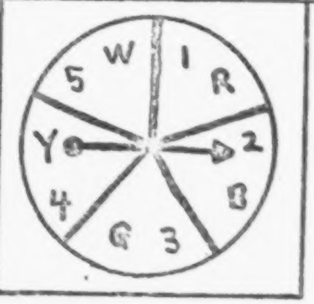
P(Y) = ?

The event (R or G) can occur twice out of 5 equally likely outcomes. So P(R or G) = 2/5. Notice that P(R) = 1/5 and P(G) = 1/5. So in this case, P(R or G) = P(R) + P(G).

In the above example the pointer can not stop on both R and G. That is, the events R and G can not both occur at the same time.

Events A and B are called mutually exclusive if and only if they can not both occur at the same time.

Now suppose you spin the pointer on the dial pictured below.



P(odd) = P(1 or 3 or 5) = 3/5.

P(even) = ?

The event (B or even) can occur twice out of five possible outcomes. So P(B or even) = 2/5. But, P(B) = 1/5 and P(even) = 2/5.

So in this case P(B or even) ≠ P(B) + P(even).

In the example just considered the pointer can stop at both B and an even number. So the events B and even are not mutually exclusive.

The two examples above illustrate the following important principle of probability.

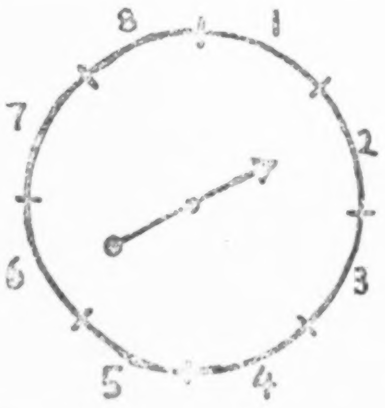
A and B are mutually exclusive events if and only if
$P(A \text{ or } B) = P(A) + P(B)$.

This result can be generalized to more than two events as in the following statement.

$E_1, E_2, \ldots, E_n$ are pairwise mutually exclusive events
(i.e. E_i and E_j are mutually ecclusive for $i \neq j$.)
if and only if $P(E_1 \text{ or } E_2 \text{ or} \cdots E_n) = P(E_1) + P(E_2) + \quad + P(E_n)$.

Exercises

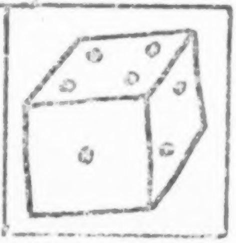
1.



a. $P(2) = ?$
b. $P(7) = ?$
c. $P(\text{even}) = ?$
d. $P(\text{odd}) = ?$
e. $P(< 3) = ?$
f. $P(\geq 6) = ?$
g. $P(< 1) = ?$
h. $P(\text{between } 0 \text{ and } 9) = ?$

2. Suppose a single die is rolled.

a. $P(1 \text{ or } 6) = ?$
b. $P(\text{even}) = ?$
c. $P(\text{not } 1) = ?$
d. $P(\text{odd}) = ?$
e. $P(\text{not } 1 \text{ or } 6) = ?$



3. Suppose a coin is tossed twice.

a. $P(HT) = ?$ (HT = head 1st toss, tail 2nd toss.)

b. $P(HH) = ?$

c. $P(TH) = ?$

d. $P(TT) = ?$

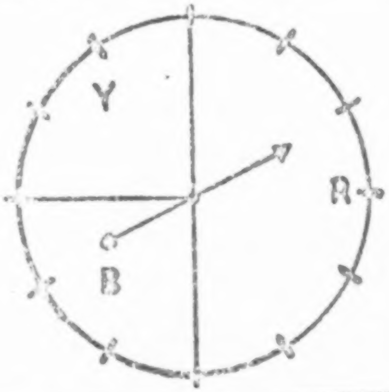
e. Are HT and TH mutually exclusive events?

f. Are HH and TT mutually exclusive events?

g. P(at least one head) = ?

h. P(at least one tail) = ?

4.



a. $P(R) = ?$

b. $P(B) = ?$

c. $P(Y) = ?$

d. $P(5) = ?$

e. P(even) = ?

f. P(odd) = ?

g. P(R or odd) = ?

h. P(B or even) = ?

i. P(B or ≤ 3) = ?

j. P(Y or ≤ 6) = ?

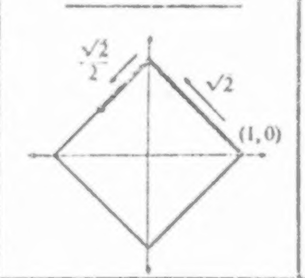
k. Which pairs of events in a through j are mutually exclusive?

Focus Passage FP12

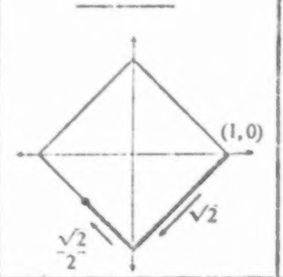
Extending the abs and ord functions

Notice that since the time of motion of the particle along the square is measured by nonnegative numbers, the domain of abs and ord is the set of nonnegative numbers. We can extend these functions so that their domains will include the negative real numbers by considering the direction of motion of the particle. Use positive numbers to measure the time of motion when the particle moves in the counterclockwise direction, and use negative numbers to measure the time of motion when the particle moves in the clockwise direction.

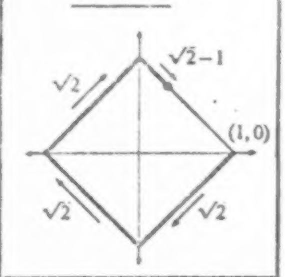
counterclockwise



clockwise



clockwise



$\text{abs}\left(\frac{3}{2}\sqrt{2}\right) = -\frac{1}{2}$	$\text{abs}\left(-\frac{3}{2}\sqrt{2}\right) = -\frac{1}{2}$	$\text{abs}\,(1 - 4\sqrt{2}) = \frac{2 - \sqrt{2}}{2}$
$\text{ord}\left(\frac{3}{2}\sqrt{2}\right) = \frac{1}{2}$	$\text{ord}\left(-\frac{3}{2}\sqrt{2}\right) = -\frac{1}{2}$	$\text{ord}\,(1 - 4\sqrt{2}) = \frac{\sqrt{2}}{2}$

To find the value of abs [or ord] for a given argument *k*, let the particle start at the point (1, 0) and move along the square at the rate of 1 unit per second for *k* seconds. The direction of motion is counterclockwise if the argument is positive and clockwise if the argument is negative. The value of abs [or ord] is the first [or second] coordinate of the final position.

[illegible]

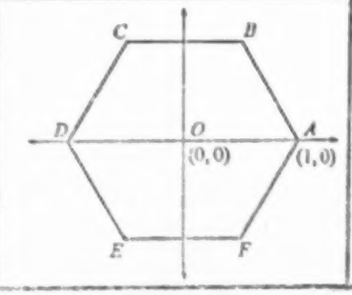
1. Compute.

(*a*) $\text{abs}\,(-\sqrt{2})$	(*b*) $\text{ord}\,(-\sqrt{2})$	(*c*) $\text{abs}\,(-1)$
(*d*) $\text{ord}\,(-1)$	(*e*) $\text{abs}\,(-5\sqrt{2})$	(*f*) $\text{ord}\,(-6\sqrt{2})$

2. Sketch graphs of abs and ord for the argument-interval from −10 to 10.

[illegible]

3. Place a regular hexagon with radius 1 on a coordinate plane in such a way that its center has coordinates (0, 0) and one of its vertices has coordinates (1, 0). Let a particle *P* move on the hexagon starting in position *A* and either counterclockwise or clockwise at the constant rate of 1 unit per second. Sketch a graph of the function which relates the first coordinate of each position to the [illegible] argument-interval from −12 to 12. Tell the domain and the range of this function. Is the function periodic? If so, what is a period of the function?



4. [illegible] coordinates (0, 0) and whose vertex *A* has coordinates (1, 0).

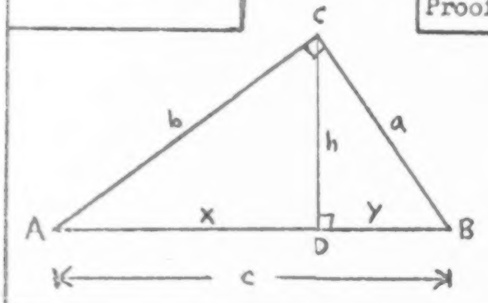
(Reprinted by permission. Algebra with Trigonometry, Fehr, et al., D. C. Heath and Company, 1963.)

Contrived Passage CP10b

Mean Proportional and the Pythagorean Theorem

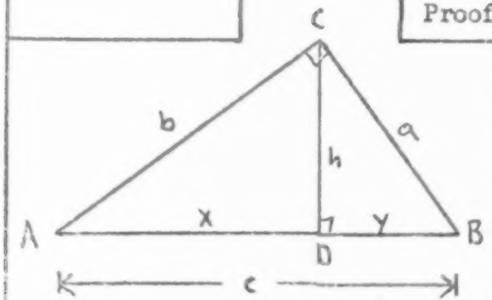
Definition 1: Given segments $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$. Then $\overline{CD}$ is the mean proportional between $\overline{AB}$ and $\overline{EF}$ if and only if $\frac{AB}{CD} = \frac{CD}{EF}$.

Theorem 1: The altitude to the hypotenuse of a right triangle is a mean proportional between the segments into which the foot of the altitude divides the hypotenuse.



Proof: Since $ADC \leftrightarrow CDB$ is a similarity it follows by definition of similarity that $\frac{AD}{CD} = \frac{DC}{DB}$ and the proof is complete.

Theorem 2: The sum of the squares of the measures of the legs of a right triangle is the square of the measure of the hypotenuse.



Proof: Since $ACD \leftrightarrow ABC$ is a similarity we have $\frac{AD}{AC} = \frac{AC}{AB}$ or $\frac{x}{b} = \frac{b}{c}$. So, $b^2 = x \cdot c$. Similarly, $CBD \leftrightarrow ABC$ is a similarity and $\frac{DB}{CB} = \frac{CB}{AB}$ or $\frac{y}{a} = \frac{a}{c}$. So, $h^2 = y \cdot c$. Thus,

$$a^2 + b^2 = x \cdot c + y \cdot c = (x + y) \cdot c = c^2 .$$

Theorem 2 is an important theorem in geometry. It is called the Pythagorean Theorem.

Exercises

1. For right ∠ ABC with hypotenuse c and legs a and b find the measure of the third side.

   a. $a = 2$, $b = 5$.

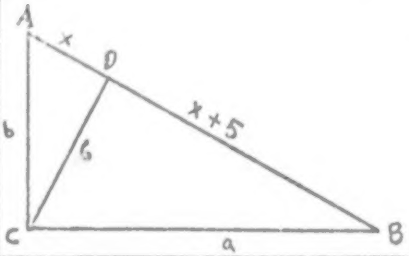
b. $a = 9$, $b = 12/2$.

c. $a = 3/4$, $c = 2$.

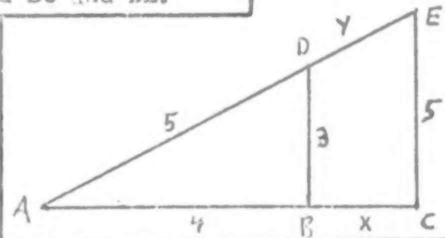
d. $b = 2/3$, $c = 3/2$.

e. $a = 3$, $b = \sqrt{a}$.

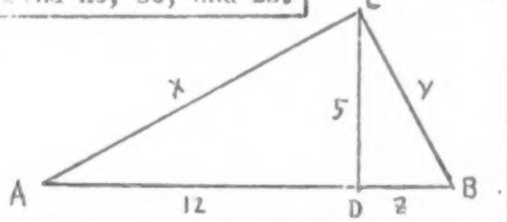
2. Find BC, AD, and BD.


A
x
D
x+5
b
6
C
a
B

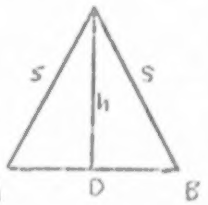
3. Find BC and DE.


E
Y
D
5
5
3
A
4
B
X
C

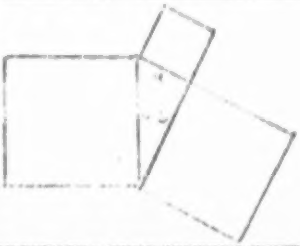
4. Find AC, BC, and DB.


C
X
Y
5
A
12
D
z
B

5. Find the altitude of an equalateral triangle in terms of the side length s.



6. Interpret the Pythagorean Theorem in terms of area.

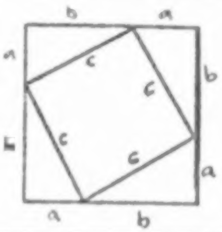


What are the areas of the squares?

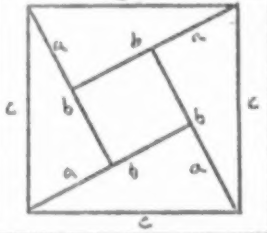
How are the areas of the squares related?

Why?

7. Use the figure to give an area proof of the Pythagorean Theorem.

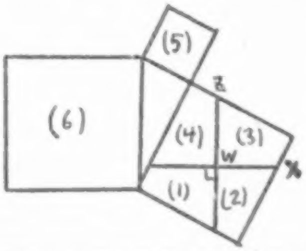


8. Use the figure to give an area proof of the Pythagorean Theorem.



9. Let W be the point of intersection of the diagonals of the square. Let $\overline{XZ}$ be parallel to the hypotenuse of the right triangle. Let $\overline{WY} \perp \overline{XZ}$.

a. How can square (6) be disected into five regions four of which are congruent to (1), (2), (3), and (4) and one which is congruent to (5). (You can cut regions congruent to (1) through (5) out of some paper and try to fit them into region (6) if you have trouble.)



b. Explain how this proves the Pythagorean Theorem.

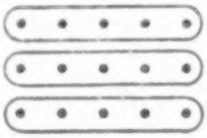
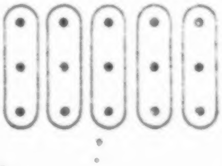
APPENDIX C

VALIDITY FOCUS PASSAGES

Focus Passage FP1

Division and sets

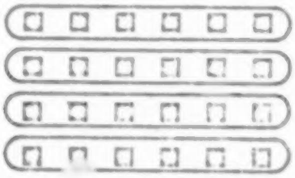
Here is a way to think about division.

Think: There are 3 sets of 5 in a set of 15. Write: $15 \div 5 = 3$ 	Think: There are 5 sets of 3 in a set of 15. Write: $15 \div 3 = 5$ 
5 is the divisor. 3 is the quotient.	3 is the divisor. 5 is the quotient.

EXERCISES

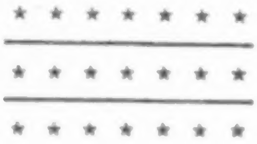
1. Draw a set of 18 dots on your paper. Ring as many sets of 6 as you can.
   [A] How many did you find? [B] Solve the equation, $18 \div 6 = \boxed{n}$.
2. Draw a set of 24 dots on your paper. Ring as many sets of 4 as you can. Solve the equation, $24 \div 4 = \boxed{n}$.
3. Draw a set of 32 dots on your paper. Ring as many sets of 8 as you can. Solve the equation, $32 \div 8 = \boxed{n}$.
4. Study the sets. Then solve the equation.

[A]



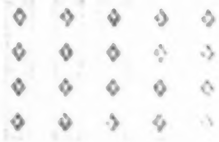
$24 \div 6 = \boxed{n}$

[B]



$21 \div 7 = \boxed{n}$

[C]



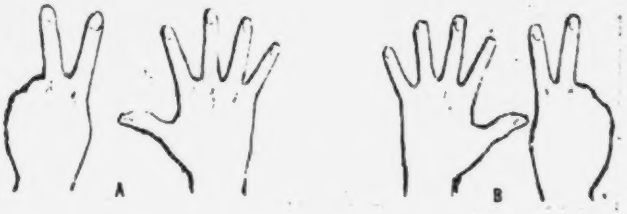
$20 \div 4 = \boxed{n}$

5. Give the missing numbers.
   [A] There are ☐ sixes in 36. $36 \div 6 = \boxed{n}$
   [B] There are ☐ sevens in 28. $28 \div 7 = \boxed{n}$

(Reprinted by permission. Elementary School Mathematics, Book 4, Eicholz and O'Daffer, Addison-Wesley Publishing Company, 1968.)

THE COMMUTATIVE PROPERTY OF ADDITION

1. a. How many fingers are shown in A? How many fingers are shown in B?



b. A number sentence for A is $2 + 5 = 7$. Write a number sentence for B.

c. Is $2 + 5 = 5 + 2$ a true sentence?

2. Solve.

a. $3 + 6 = n$ **b.** $6 + 3 = n$ **c.** $3 + 6 = n + 3$
d. $12 + 17 = n$ **e.** $17 + 12 = n$ **f.** $n + 17 = 17 + 12$

Two whole numbers may be added in either order without affecting the sum. This is the **commutative property of addition.**

3. Add. Why are these sums the same? We check by adding in the opposite direction.

$$\begin{array}{r}34\\+25\\\hline\end{array} \qquad \begin{array}{r}25\\+34\\\hline\end{array}$$

EXERCISES

Solve.

1. $7+9=9+\square$ **2.** $47+3=n+47$ **3.** $\square+10=10+8$

4. $n+94=94+38$ **5.** $86+48=n+86$ **6.** $13+n=29+13$

Add. Check by adding in the opposite direction.

7. $\begin{array}{r}41\\+28\\\hline\end{array}$ **8.** $\begin{array}{r}126\\+63\\\hline\end{array}$ **9.** $\begin{array}{r}34\\+52\\\hline\end{array}$ **10.** $\begin{array}{r}493\\+5\\\hline\end{array}$

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Focus Passage FP5

74 **Exploring Ideas**

Multiplication of rational numbers

In unit 6 you learned that multiplication of natural numbers is a mapping of the set of ordered pairs of natural numbers onto the set of natural numbers. You also learned that the product of two natural numbers is a natural number. In this lesson you will learn how to find the product of two rational numbers.

A Look at D1. What are the first and second components of each ordered pair of rational numbers named in the display?

B $(\frac{2}{3}, \frac{3}{4})$ is mapped onto the product $\frac{2}{3} \times \frac{3}{4}$. Describe the other mappings represented in D1.

C Read sentence A in D2. Sentence A expresses a true statement about the product of the rational numbers $\frac{2}{3}$ and $\frac{3}{4}$. The number $\frac{6}{12}$ is the product of $\frac{2}{3}$ and $\frac{3}{4}$. Is 6 the product of the numerators of $\frac{2}{3}$ and $\frac{3}{4}$? Is 12 the product of the denominators of $\frac{2}{3}$ and $\frac{3}{4}$? Remember that when we mention numerators and denominators, we are referring to the fractions that indicate the rational numbers.

D Sentence B also expresses a true statement about the product of two rational numbers. What are these numbers? Is $1\frac{6}{8}$ their product?

The *product of two rational numbers* is a rational number indicated by a fraction. The numerator of the fraction is the product of the two numerators, and the denominator is the product of the two denominators. The work that follows will help you use variables to develop the definition of the product of two rational numbers.

Product of rational numbers: closure property of multiplication;
well-defined property of multiplication

$$\begin{tikzcd}
\left(\frac{2}{3}, \frac{3}{4}\right) \arrow[d] & \left(\frac{8}{10}, \frac{2}{1}\right) \arrow[d] & \left(\frac{4}{5}, \frac{5}{4}\right) \arrow[d] & \left(\frac{32}{3}, \frac{0}{8}\right) \arrow[d] \\
\frac{2}{3} \times \frac{3}{4} & \frac{8}{10} \times \frac{2}{1} & \frac{4}{5} \times \frac{5}{4} & \frac{32}{3} \times \frac{0}{8}
\end{tikzcd}$$

D1

A $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$.

B $\frac{8}{10} \times \frac{2}{1} = \frac{16}{10}$.

D2

C $\frac{2}{3} \times \frac{3}{4} = \frac{2 \cdot 3}{3 \cdot 4}$.

D $\frac{8}{10} \times \frac{2}{1} = \frac{8 \cdot 2}{10 \cdot 1}$.

E $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

D3

Universe for a and $c = N$.
Universe for b and $d = C$.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

F $\frac{5}{8} \times \frac{1}{5} = \frac{5 \times 1}{8 \times 5}$.

G $\frac{7}{6} \times \frac{2}{3} = \frac{7 \times 2}{6 \times 3}$.

H $\frac{75}{1} \times \frac{1}{4} = \frac{75 \times 1}{1 \times 4}$.

I $\frac{3}{4} \times \frac{4}{3} = \frac{3 \times 4}{4 \times 3}$.

D4

product of two rational numbers. The universe for a and c is N. The universe for b and d is C. For each replacement of a, b, c, and d, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

E Look at D3. Does sentence C express a true statement? Does sentence D express a true statement? Explain your answers.

F Now look at sentence E in D3. Think of $\frac{a}{b}$ and $\frac{c}{d}$ as any two rational numbers. The universe for a and c is N. The universe for b and d is C. Is the numerator of $\frac{ac}{bd}$ the product of a and c? Is the denominator of $\frac{ac}{bd}$ the product of b and d?

Now we can define the product, $\frac{a}{b} \times \frac{c}{d}$, of the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$. The universe for a and c is N. The universe for b and d is C. For each replacement of a, b, c, and d, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

G Read sentence F in D4. What replacements were made for the variables in $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ to obtain the statement expressed by sentence F? Is the statement true? How do you know?

H What replacements were made for the variables in $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ to obtain the statement expressed by sentence G? By sentence H? By sentence I? Is each of the statements true?

Decide whether each of the following sentences expresses a true statement or a false statement.

I $\frac{7}{8} \times \frac{1}{4} = \frac{7}{32}$. **K** $\frac{16}{5} \cdot \frac{3}{10} = \frac{48}{50}$. **M** $\frac{4}{3} \cdot \frac{1}{9} = \frac{5}{12}$.
J $\frac{2}{3}(\frac{1}{2}) = \frac{4}{3}$. **L** $\frac{0}{6} \times \frac{6}{1} = \frac{6}{6}$. **N** $\frac{11}{12}(\frac{12}{11}) = \frac{132}{132}$.

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Seeing Through Mathematics,
Book One, Van Engen, et al.,
Scott, Foresman and Company.)

Focus Passage FP8

Expressions with Grouping Symbols

In the preceding exercises you used the distributive law to get a simpler algebraic expression. For example, $2x - 3(2x + 5) = 2x - 6x - 15 = -4x - 15$. The expression $2x - 3(2x + 5)$ was changed to $-4x - 15$. Note that the new expression no longer contains parentheses as grouping symbols. You can change an expression with parentheses to one without them by using the distributive law.

EXAMPLES

Rewrite without parentheses and combine like terms.

1: $6 - 4(2x - 5) = 6 - 8x + 20 = 26 - 8x$

2: $8 - (3x - 5)$ Think: $8 - 1(3x - 5)$

$8 - (3x - 5) = 8 - 3x + 5 = 13 - 3x$

3: $6 + (9 - 7x)$ Think: $6 + 1(9 - 7x)$

$6 + (9 - 7x) = 6 + 9 - 7x = 15 - 7x$

Brackets [ ] are used just as parentheses to group numbers and algebraic expressions. When an expression has two or more pairs of grouping symbols, it is usually easier to begin by removing the innermost pair. The following example shows how to remove grouping symbols and then to combine like terms.

4:

$$\begin{aligned} 3x - 4[2x - 5(x - 2)] &= 3x - 4[2x - 5x + 10] && \text{Removing ( )} \\ &= 3x - 4[-3x + 10] && \text{Combining terms in [ ]} \\ &= 3x + 12x - 40 && \text{Removing [ ]} \\ &= 15x - 40 && \text{Combining terms} \end{aligned}$$

EXERCISES

Rewrite the following without grouping symbols and combine like terms:

1. $9 - (7a - 3)$
2. $9 - (7a + 3)$
3. $9 - (-7a - 3)$
4. $9 + (7a - 3)$
5. $7a + (2a - 3)$
6. $7a - (+2a - 3)$
7. $7a - (2a - 3)$
8. $7a - (2a + 3)$
9. $7a - (-2a + 3)$
10. $7a + (-2a + 3)$
11. $x - (3 + 2x)$
12. $x - (-3 + 2x)$

13. $x - (-3 - 2x)$
14. $(a + b) - b$
15. $-(a - b) + b$
16. $-(a - b) - b$
17. $5 + (2x - 3)$
18. $2a + (a - 7) + 9$
19. $2x + 2[x + 3(x - 5)]$
20. $5[x - 8(x - 2)] + 10x$
21. $7y - 4[y + 2(y - 8)]$
22. $-6[7m - 2(6m - 9)] - 8m$
23. $8[3x - 2(x - 5)]$
24. $9[3x - 7(x - 9)] - 12x$

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Focus Passage FP11

7.6. *Fractional Exponents.* Let x be a positive number. To take the $\sqrt{x^6}$, we should think, "What is one of the two equal factors having a product of x^6?"

Since $\quad x^3 \cdot x^3 = x^6, \quad \sqrt{x^6} = x^3.$

Also $\quad x^2 \cdot x^2 = x^4 \quad \sqrt{x^4} = x^2$

$$x \cdot x = x^2 \qquad \sqrt{x^2} = x$$

$$x^{(?)} \cdot x^{(?)} = x \qquad \sqrt{x} = x^{(?)}.$$

To indicate $\sqrt{x}$ using an exponent we must find an exponent such that when it is added to itself, the sum equals one. This exponent is $\frac{1}{2}$.

$$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x \qquad \sqrt{x} = x^{\frac{1}{2}}$$

$$x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x \qquad \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$x^{\frac{1}{4}} \cdot x^{\frac{1}{4}} \cdot x^{\frac{1}{4}} \cdot x^{\frac{1}{4}} = x \qquad \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$x^{\frac{2}{3}} \cdot x^{\frac{2}{3}} \cdot x^{\frac{2}{3}} = x^2 \qquad \sqrt[3]{x^2} = x^{\frac{2}{3}}$$

$$x^{\frac{3}{5}} \cdot x^{\frac{3}{5}} \cdot x^{\frac{3}{5}} \cdot x^{\frac{3}{5}} \cdot x^{\frac{3}{5}} = x^3 \qquad \sqrt[5]{x^3} = x^{\frac{3}{5}}$$

We see from these examples that we may define $x^{\frac{a}{b}}$ as $\sqrt[b]{x^a}$, if $b \neq 0$.

Consider the domain of each of the variables in the following examples and exercises as the set of positive real numbers.

Examples

1. Write the following radicals by the use of fractional exponents:

   (a) $\sqrt[3]{xy^2}$ (b) $2\sqrt[5]{yz^2}$ (c) $x^3\sqrt{x}$

2. Write with radicals in simplest form:

   (a) $2x^{\frac{1}{2}}$ (b) $a^{\frac{1}{4}}b^{\frac{3}{4}}$ (c) $(ab)^{-\frac{1}{2}}$

Answers:

1. (a) $\sqrt[3]{xy^2} = x^{\frac{1}{3}}y^{\frac{2}{3}}$ (b) $2\sqrt[5]{yz^2} = 2y^{\frac{1}{5}}z^{\frac{2}{5}}$ (c) $x^3\sqrt{x} = x^3x^{\frac{1}{2}} = x^{\frac{7}{2}}$

2. (a) $2x^{\frac{1}{2}} = 2\sqrt{x}$ (b) $a^{\frac{1}{4}}b^{\frac{3}{4}} = \sqrt[4]{ab^3}$ (c) $(ab)^{-\frac{1}{2}} = \dfrac{1}{\sqrt{ab}} = \dfrac{\sqrt{ab}}{ab}$

Exercises

Write with fractional exponents:

1. $\sqrt[3]{x^2}$ 2. $\sqrt[4]{y^3}$ 3. $\sqrt[5]{x}$ 4. $\sqrt[6]{a}$ 5. $2\sqrt{w^3}$

6. $5\sqrt{b}$ 7. $\sqrt{xy}$ 8. $\sqrt{ab}$ 9. $\sqrt{x^2y^3}$ 10. $\sqrt{xy^2}$

11. $\sqrt[3]{xy^2}$ 12. $\sqrt{a^2b^3}$ 13. $x\sqrt{y}$ 14. $a\sqrt{b}$ 15. $x^2\sqrt{x}$

16. $a\sqrt{a}$ 17. $\sqrt{bx}$ 18. $\sqrt[4]{ay^2}$

Write with radicals:

19. $x^{\frac{1}{2}}$ 20. $y^{\frac{2}{3}}$ 21. $a^{\frac{1}{4}}$ 22. $c^{\frac{3}{5}}$ 23. $2a^{\frac{1}{3}}$

24. $3b^{\frac{2}{3}}$ 25. $(2x)^{\frac{1}{2}}$ 26. $(3y)^{\frac{1}{4}}$ 27. $x^{\frac{1}{2}}y^{\frac{1}{2}}$ 28. $a^{\frac{1}{3}}b^{\frac{1}{3}}$

29. $2^{\frac{1}{3}}x^{\frac{1}{3}}$ 30. $5^{\frac{1}{3}}y^{\frac{1}{3}}$ 31. $xy^{\frac{1}{3}}$ 32. $ab^{\frac{1}{2}}$ 33. $x^{\frac{1}{2}}y^{\frac{1}{3}}$

34. $a^{\frac{2}{3}}b^{\frac{1}{3}}$ 35. $w^{\frac{1}{3}}y^{\frac{1}{4}}$ 36. $x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{3}{5}}$

Write with radicals and simplify:

37. $x^{-\frac{1}{2}}$ 38. $a^{-\frac{2}{3}}$ 39. $bc^{-\frac{1}{2}}$ 40. $am^{-\frac{1}{2}}$ 41. $(ab)^{-\frac{1}{2}}$

42. $(xy)^{-\frac{1}{3}}$ 43. $(a^3b)^{-\frac{1}{4}}$ 44. $2a^{-\frac{1}{3}}$ 45. $5b^{-\frac{1}{4}}$ 46. $a^{\frac{3}{2}}$

47. $b^{\frac{5}{3}}$ 48. $a^{\frac{1}{2}}b^{\frac{3}{2}}$ 49. $x^{\frac{5}{3}}y^{\frac{1}{3}}$ 50. $2a^{\frac{3}{2}}$ 51. $4b^{\frac{3}{2}}$

52. $(a^3)^{-\frac{1}{2}}$ 53. $(b^2)^{-\frac{1}{4}}$ 54. $a^{\frac{3}{2}}b^{-\frac{1}{2}}$ 55. $x^{\frac{1}{2}}y^{-\frac{3}{2}}$ 56. $a^2b^{\frac{1}{2}}c^{-\frac{1}{2}}$

Find the value of the following:

57. $4^{\frac{1}{2}}$ 58. $9^{-\frac{1}{2}}$ 59. $16^{\frac{3}{4}}$ 60. $8^{\frac{1}{3}}$ 61. $25^{-\frac{1}{2}}$

62. $(3^{\frac{1}{2}})^{-2}$ 63. $(4^{\frac{1}{3}})^{-3}$ 64. $(0.008)^{\frac{1}{3}}$ 65. $(0.25)^{\frac{1}{2}}$ 66. $(\frac{9}{4})^{-\frac{1}{2}}$

67. $(\frac{1}{8})^{-\frac{2}{3}}$

Simplify:

◆68. $[(a^{n+1})(a^{n-1})]^{\frac{1}{2}}$ ◆69. $(b^{\frac{x}{y}})^{2y}$ ◆70. $(a^{\frac{2}{x}})^{\frac{x}{2}}$

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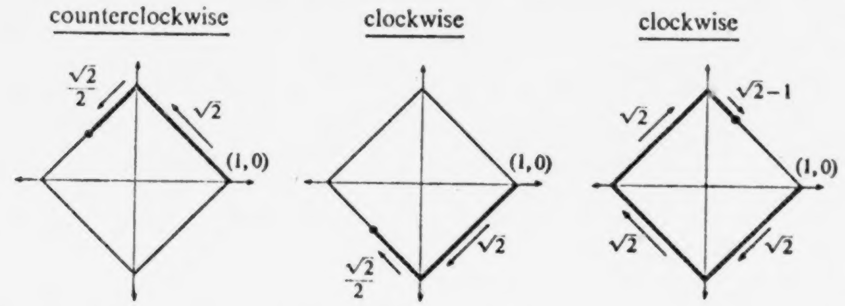
Focus Passage FP12

Extending the abs and ord functions

Notice that since the time of motion of the particle along the square is measured by nonnegative numbers, the domain of abs and ord is the set of nonnegative numbers. We can extend these functions so that their domains will include the negative real numbers by considering the direction of motion of the particle. Use positive numbers to measure the time of motion when the particle moves in the counterclockwise direction, and use negative numbers to measure the time of motion when the particle moves in the clockwise direction.



counterclockwise

clockwise

clockwise

$$\operatorname{abs}\left(\frac{3}{2}\sqrt{2}\right) = -\frac{1}{2} \qquad \operatorname{abs}\left(-\frac{3}{2}\sqrt{2}\right) = -\frac{1}{2} \qquad \operatorname{abs}(1 - 4\sqrt{2}) = \frac{2-\sqrt{2}}{2}$$

$$\operatorname{ord}\left(\frac{3}{2}\sqrt{2}\right) = \frac{1}{2} \qquad \operatorname{ord}\left(-\frac{3}{2}\sqrt{2}\right) = -\frac{1}{2} \qquad \operatorname{ord}(1 - 4\sqrt{2}) = \frac{\sqrt{2}}{2}$$

To find the value of abs [or ord] for a given argument k, let the particle start at the point (1, 0) and move along the square at the rate of 1 unit per second for k seconds. The direction of motion is counterclockwise if the argument is positive and clockwise if the argument is negative. The value of abs [or ord] is the first [or second] coordinate of the final position.

PRACTICE EXERCISES

▼

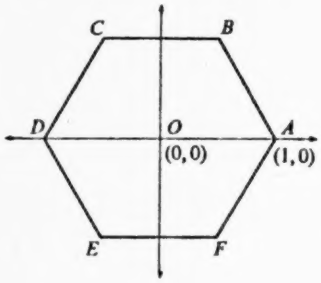
1. Compute.

   (*a*) abs $(-\sqrt{2})$ (*b*) ord $(-\sqrt{2})$ (*c*) abs (-1)

   (*d*) ord (-1) (*e*) abs $(-5\sqrt{2})$ (*f*) ord $(-6\sqrt{2})$

2. Sketch graphs of abs and ord for the argument-interval from -10 to 10.

▼▼



3. Place a regular hexagon with radius 1 on a coordinate plane in such a way that its center has coordinates (0, 0) and one of its vertices has coordinates (1, 0). Let a particle *P* move on the hexagon starting in position *A* and either counterclockwise or clockwise at the constant rate of 1 unit per second. Sketch a graph of the function which relates the first coordinate of each position to the time of travel to that position. [Use the argument-interval from -12 to 12.] Tell the domain and the range of this function. Is the function periodic? If so, what is a period of the function?

4. Repeat Exercise 3 for the regular octagon whose center has coordinates (0, 0) and whose vertex *A* has coordinates (1, 0).

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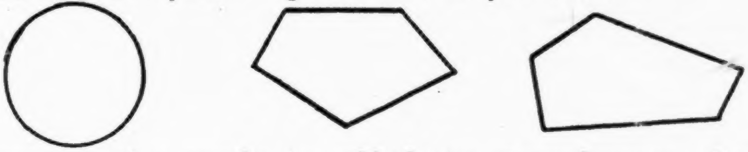
APPENDIX D

CONTRIVED PASSAGES

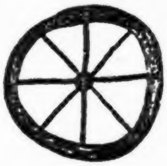
Contrived Passage CP2a

Circumference

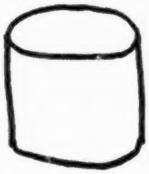
Is a circle a closed plane figure? How do you know?



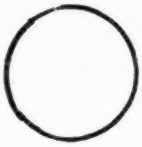
The distance around a corcle is called its circumference. Circumference is to a circle as ______?______ is to a polygon. How would you use a ruler to find the perimeter of a polygon? How would you use a tape measure to find the circumferences of the objects below? How would you find the diameter lengths?



wheel



can



plate

Copy and complete the table below. For 4 and 5 choose two circular objects and measure the circumference and diameter of each.

	Circumference	Diameter Length	$\frac{\text{Circumference}}{\text{Diameter Length}}$
Wheel	88 inches	28 inches	
Can	38 inches	12 inches	
Plate	$31\frac{3}{4}$ inches	10 inches	
(4)			
(5)			

Are the five circumference lengths in your table different? Are the diameter lengths different? How much difference is there between the numbers in the last column? Name two numbers which all the numbers in the last column are between.

It is true that when the circumference of any circle is divided by the length of a diameter of that circle, the quotient is always the same number. This number is named by the Greek letter π (pi). The number π is about $3\frac{1}{7}$. How near to $3\frac{1}{7}$ were your answers in the last column of the above table?

If C stands for the circumference of a circle and d stands for the diameter then

a. $\frac{C}{d} = \underline{\ ?\ \ }$

b. $C = \underline{\ \ ?\ \ } \cdot \underline{\ \ ?\ \ }$

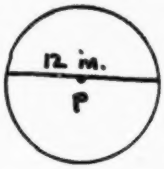
and c. $d = \frac{?}{?}$.

How can you use the length of a diameter of a circle to find the circumference? How can you use the circumference of a circle to find the length of a diameter?

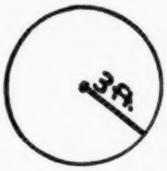
Exercises

Find the circumferences of the circles in 1 - 4.

1.



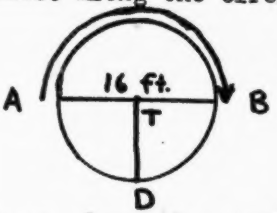
2.



3. diameter = 17 yards

4. radius = $\frac{1}{2}$ inch

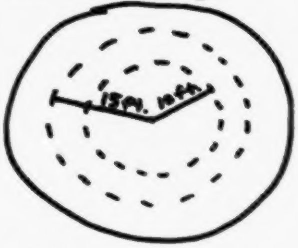
5. a. What is the distance along the circle from A to B?



b. What is the distance along the circle from A to D if you measure in the direction of the arrow ?

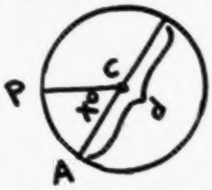
c. What is the distance from A to D if you go in the opposite direction?

6. A merry-go-round makes 30 complete turns. How many feet will Ronald and Martha each travel if Martha's horse is on the inner ring and Ronald's is on the outer ring?



7. a. What fraction of 180° is x° ?

b. What fraction of one-half of the circumference is the shortest distance along the circle from A to P?



c. Write a formula for finding the shortest distance along the circle from A to P in terms of x and the diameter d .

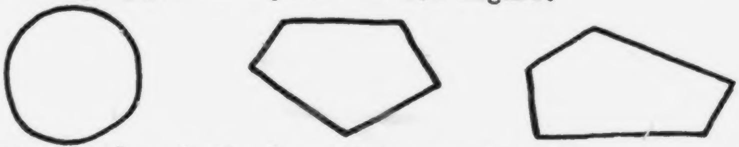
d. Write a formula for finding the longest distance along the circle from A to P in terms of x and the diameter d .

e. If $x^\circ = 60^\circ$ and d = 5 inches what is the shortest distance along the circle from A to P? What is the longest such distance?

Contrived Passage CP2b

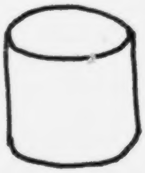
Circumference

A circle like a ploygon is a plane closed figure.

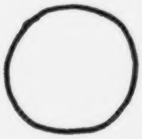


The distance around a circle is called its circumference. Circumference is to a circle as perimeter is to a polygon. You could use a ruler to find the length of each side of a polygon and then add these lengths to find the perimeter of the polygon. You could use a tape measure to find the circumferences and diameter lengths of the objects below.

wheel



can



plate

The table below gives the circumference, diameter length, and quotient of circumference divided by diameter length of each of these objects.

	Circumference	Diameter Length	Circumference / Diameter Length
Wheel	88 inches	28 inches	3 1/7
Can	38 inches	12 inches	3 1/6
Plate	31$\frac{3}{4}$ inches	10 inches	3 7/40

Notice that while the circumferences differ and the diameter lengths differ, the quotients of circumference divided by diameter length do not differ very much. It is only because of inaccuracy of measurement that the numbers in the last column differ at all!

It is true that when the circumference of any circle is divided by the length of a diameter of that circle, the quotient is always the same number. This number is named by the Greek letter π (pi). The number π is about $3\frac{1}{7}$. Notice how close to $3\frac{1}{7}$ each of the numbers in the final column of the above table is.

If C stands for the circumference of a circle and d stands for the diameter then

a. $\frac{C}{d} = \underline{\quad ? \quad}$

b. $C = \underline{\quad ? \quad} \cdot \underline{\quad ? \quad}$

and c. $d = \frac{?}{?}$.

You can use formula (b) with $3\frac{1}{7}$ for π to find the circumference of a circle when you know the length of a diameter. You can use formula (c) to find the length of a diameter of a circle when you know the circumference.

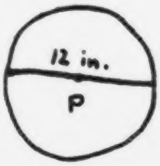
Exercises

1. Complete the table.

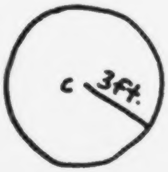
Circumference	Diameter	Radius
	5 inches	
		4 feet
39 inches		
	d	
		r
C		

Find the circumference of each circle in 2 through 9.

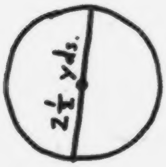
2.



3.



4.



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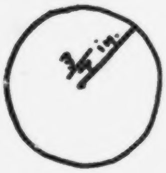


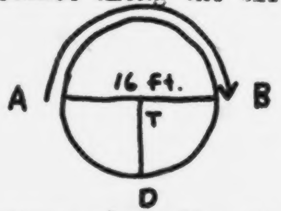
6. d = 17 yards

7. r = ½ inch

8. d = 7/11 inch

9. r = 3/4 mile

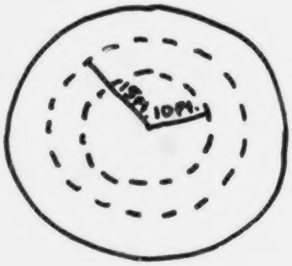
10. a. What is the distance along the circle from A to B?



b. What is the distance along the circle from A to D if you travel in the direction of the arrow?

c. What is the distance from A to D if you travel in the opposite direction?

11. A merry-go-round makes 30 complete turns. How many feet will Ronald and Martha each travel if Marthas horse is on the inner ring and Ronald's is on the outer ring?



Contrived Passage CP4a

Using Equations and Inequalities

A 12 foot board is cut into 3 pieces of equal length. To find the length of each piece, consider the equation

$$3 \cdot x = 12 .$$

Solving this equation we find, $x = 4$. Hence, each piece is 4 feet long.

Five girls want to share 23 apples evenly. To find the greatest number of whole apples each girl can get, consider the inequality

$$n \cdot 5 < 23 .$$

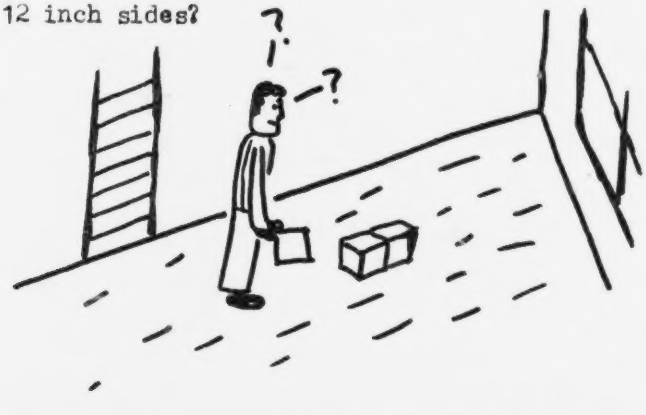
Using the replacement set of whole numbers, the solution set is $\{0, 1, 2, 3, 4\}$. So the greatest number of apples each girl can get is 4.

Exercises

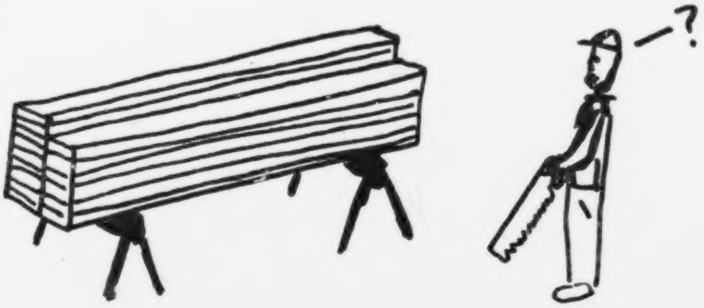
Write and solve equations or inequalities to answer the problems.

1. In 3 days the liner on which Carol was traveling steamed 1536 miles. If it steamed the same distance each day then how far did it go each day?

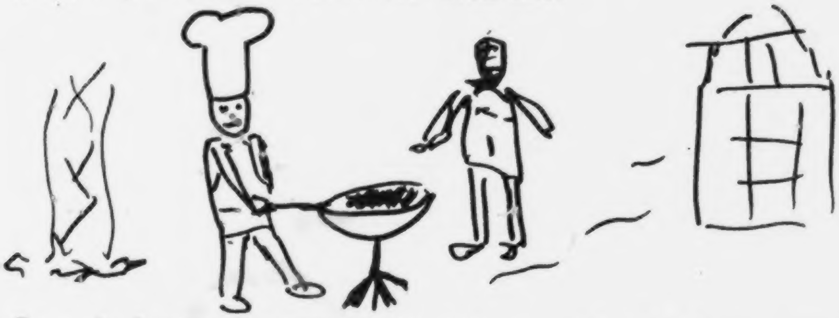
2. How many tile would it take to cover the floor of a rectangular room with length 20 feet and width 13 feet if the tile are square with 12 inch sides?



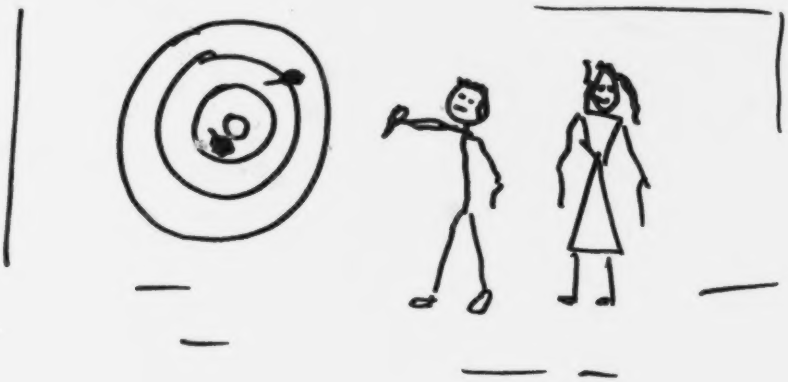
3. How many 5 foot boards can be made out of 17 boards of length 12 feet?



4. If you bought 25 hotdogs for a cook-out and each person at the cook-out has to have 3 hotdogs, what is the greatest number of people you could have at the cook-out?



5. In a dart game Sam scored 25, 10, 25, and 35. Penny scored 25, 25, 15, and 20. Who won the game and by how much?



Contrived Passage CP4b

Using Equations and Inequalities

Equations and inequalities can be used to solve real world problems. To do this follow the steps outlined below.

1. Express the unknown quantity or what you want to find as some symbol, say x.
2. Using the information given write an equation such as

   $$a + x = b$$

   or an inequality such as

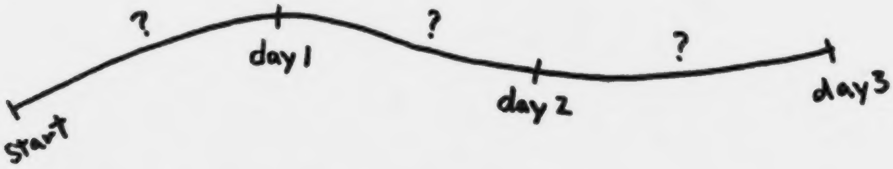
   $$x \cdot c < d$$

   relating x to known quantities a, b, c, and d.
3. Solve the equation or inequality to find the values of x and thus the answer to the problem.

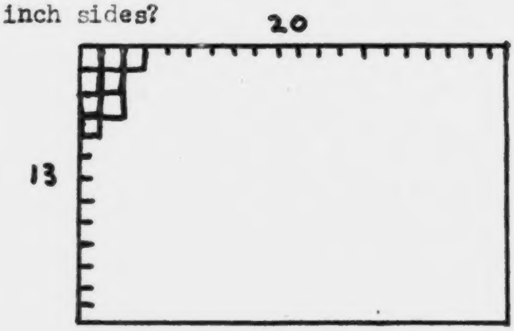
Exercises

Write and solve equations or inequalities to answer the problems.

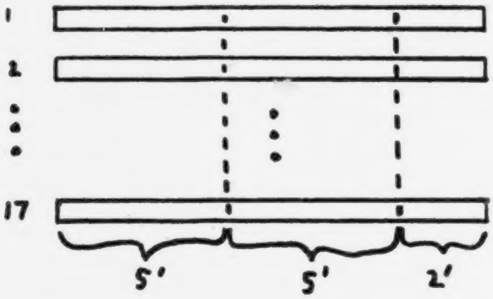
1. In 3 days the liner on which Carol was traveling steamed 1536 miles. If it steamed the same distance each day how far did it go each day?



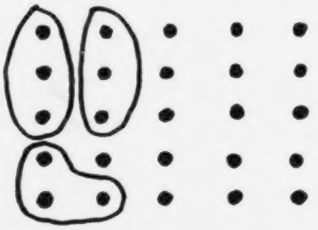
2. How many tile would it take to cover the floor of a rectangular room with length 20 feet and width 13 feet if the tile are square with 12 inch sides?



3. How many 5 foot boards can be made out of 17 boards of length 12 feet?



4. If you bought 25 hotdogs for a cook-out and each person at the cook-out has to have 3 hotdogs, what is the greatest number of people you could have at the cook-out?



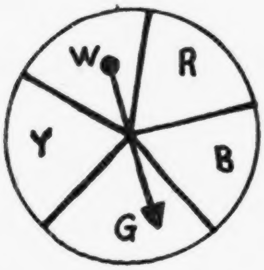
5. In a dart game Sam scored 25, 10, 25, and 35. Penny scored 25, 25, 15, and 20. Who won the game and by how much?

Sam	Penny
25	25
10	25
25	15
35	20

Contrived Passage CP6a

Probability of A or B

Spin the pointer on the dial below.



R = red

B = blue

G = green

Y = yellow

W = white

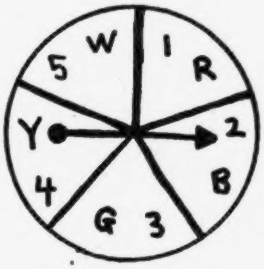
P(B) = 1/5

P(Y) = ?

The event (R or G) can occur twice out of 5 equally likely outcomes. So P(R or G) = 2/5. Notice that P(R) = 1/5 and P(G) = 1/5. So in this case, P(R or G) = P(R) + P(G).

In the above example the pointer can not stop on both R and G. That is, the events R and G can not both occur at the same time. Events A and B are called mutually exclusive if and only if they can not both occur at the same time.

Now suppose you spin the pointer on the dial pictured below.



P(odd) = P(1 or 3 or 5) = 3/5.

P(even) = ?

The event (B or even) can occur twice out of five possible outcomes. So P(B or even) = 2/5. But, P(B) = 1/5 and P(even) = 2/5. So in this case P(B or even) ≠ P(B) + P(even).

In the example just considered the pointer can stop at both B and an even number. So the events B and even are not mutually exclusive.

The two examples above illustrate the following important principle of probability.

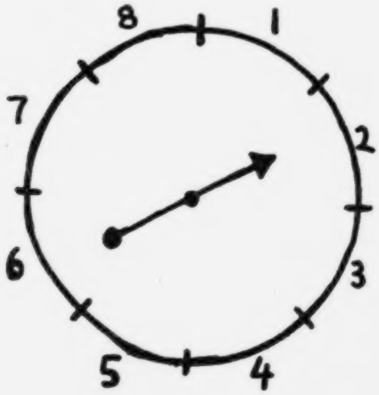
A and B are mutually exclusive events if and only if $P(A \text{ or } B) = P(A) + P(B)$.

This result can be generalized to more than two events as in the following statement.

$E_1, E_2, \dots, E_n$ are pairwise mutually exclusive events (i.e. E_i and E_j are mutually ecclusive for $i \neq j$.) if and only if $P(E_1 \text{ or } E_2 \text{ or} \cdots E_n) = P(E_1) + P(E_2) + \quad + P(E_n)$.

Exercises

1.



a. P(2) = ?

b. P(7) = ?

c. P(even) = ?

d. P(odd) = ?

e. $P(<3)$ = ?

f. $P(\geq 6)$ = ?

g. $P(<1)$ =?

h. P(between 0 and 9) = ?

2. Suppose a single die is rolled.

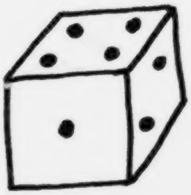
a. P(1 or 6) = ?

b. P(even) = ?

c. P(not 1) = ?

d. P(odd) = ?

e. P(not 1 or 6) = ?



3. Suppose a coin is tossed twice.

   a. P(HT) = ?   (HT = head 1st toss, tail 2nd toss.)

   b. P(HH) = ?

   c. P(TH) = ?

   d. P(TT) = ?

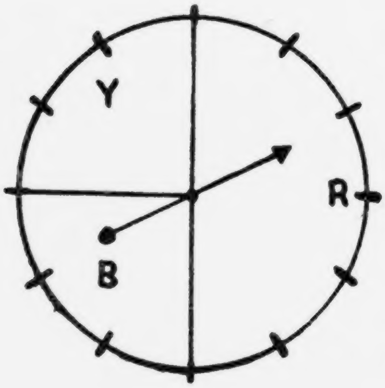
   e. Are HT and TH mutually exclusive events?

   f. Are HH and TT mutually exclusive events?

   g. P(at least one head) = ?

   h. P(at least one tail) = ?

4.



   a. P(R) = ?

   b. P(B) = ?

   c. P(Y) = ?

   d. P(5) = ?

   e. P(even) = ?

   f. P(odd) = ?

   g. P(R or odd) = ?

   h. P(B or even) = ?

   i. P(B or $\leq$ 3) = ?

   j. P(Y or $<$ 6) = ?

   k. Which pairs of events in a through j are mutually exclusive?

Contrived Passage CP6b

Probability of A or B

The pointer on a circular dial spins and stops on one of five sectors of equal area. The sectors are colored red, blue, green, yellow, and white. The probability that the pointer stops on blue is 1/5. What is the probability that the pointer stops on red?

The event the pointer stops on red or green can occur twice out of five equally likely possible outcomes. So the probability of red or green is 2/5. Notice that in this example the probability of red is 1/5 and the probability of green is also 1/5. So in this example the probability of red or green is equal to the probability of red plus the probability of green.

In the above example the pointer on the dial can not stop at both red and green. That is, the events red and green can not both occur at the same time. Two events that can not both occur at the same time are called mutually exclusive events.

Now suppose we number the colored sectors of the dial in the above example. Let the red sector be numbered one, blue be two, green be three, yellow be four, and white be five. The event the pointer stops at blue or an even number can occur twice out of five possible outcomes. That is, two out of the five possible sectors would produce the required event. So the probability of blue or even is 2/5. But, in this example the probability of blue is 1/5 and the probability of an even number is 2/5. So in this example the probability of blue or even is not equal to the sum of the probabilities of blue and even.

In the example just given the pointer can stop at both blue and an even number. That is, the events blue and even can both occur at the same time. So in the example the events are not mutually exclusive.

The two examples above illustrate the following important principle of probability.

Two events are mutually exclusive if and only if the probability of one or the other is the sum of the probabilities of each one.

This result can be generalized to more than one event as in the following statement.

N events are pairwise mutually exclusive (i.e. each pair of these events is a mutually exclusive pair.) if and only if the probability of the first or the second or ... the nth event is the sum of the probabilities of each single event.

Exercises

1. A dial is divdied into eight sectors of equal area and numbered one through eight. When the pointer is spun what is the probability of
   a. getting two?
   b. getting seven?
   c. getting an even number?
   d. getting an odd number?
   e. getting a number less than three?
   f. getting a number greater than or equal to six?
   g. getting a number less than one?
   h. getting a number between zero and nine?
2. Suppose a single die is rolled. What is the probability of
   a. getting one or six?
   b. getting an even number?
   c. getting an odd number?
   d. getting something other than one?
   e. not getting one or six?

3. Suppose a coin is tossed twice.
    - a. What is the probability of getting two heads?
    - b. Of getting a head and then a tail?
    - c. Of getting a tail and then a head?
    - d. Of getting two tails?
    - e. Are getting a head and then a tail and getting a tail and then a head mutually exclusive events?
    - f. Are getting two heads and getting two tails mutually exclusive?
    - g. What is the probability of getting at least one head?
    - h. What is the probability of getting at least one tail?

4. Suppose a circular dial is divided into twelve sectors of equal area numbered one through twelve. Suppose sectors one through six are red, sectors seven through nine are blue, and the remaining sectors are yellow. What is the probability of

    a. getting red? b. getting blue?

    c. getting yellow? d. getting five?

    e. getting an even? f. getting an odd?

    g. getting red or an odd? h. getting blue or an even

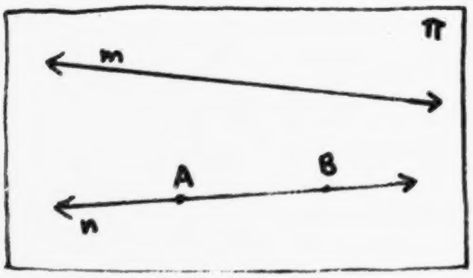
    i. getting a blue or a number less than or equal to three?

    j. getting a yellow or a number less than six?

    k. Which pairs of events in e. through j are mutually exclusive?

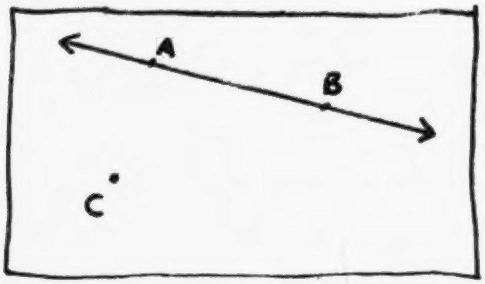
Contrived Passage CP7a

Some Logical Consequences of the Axioms



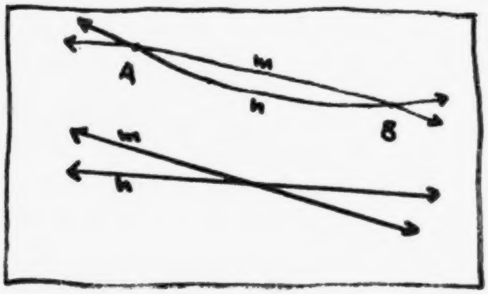
Suppose m is a line in plane π. By Axiom 1(a) there is a line n in π with $n \neq m$. By Axiom 1(b) there are distinct points A and B in line n. Now if both A and B were in m then we would have $m = n$ which is not the case. Hence, at least one of the points A or B is not in m. Thus, we have the following theorem.

Theorem 1: If m is a line in plane π then there is a point in π which is not in m.



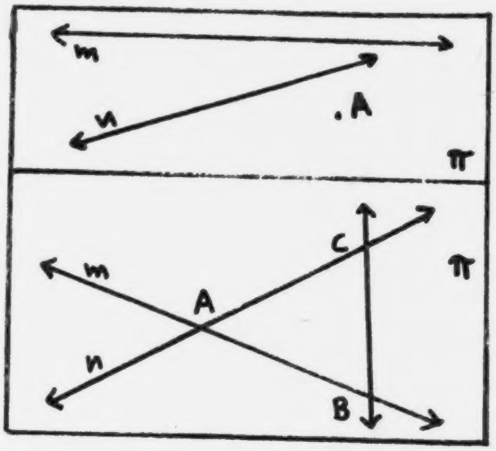
Now, by Axiom 1(a) there exists a line m in plane π. By Axiom 1(b) there are distinct points A and B on m. By Theorem 1 there exists a point C in plane π but not in line m. Hence, we have our second theorem.

Theorem 2: There are at least three non-collinear points in a given plane.



Let m and n be two distinct lines in a plane π. We know such lines exist by Axiom 1(a). Assume there are two distinct points A and B so that $A \in m \cap n$ and $B \in m \cap n$. Then m and n are two distinct lines each containing points A and B. But, this is impossible by Axiom 2. Hence, the above assumption is impossible and we have proven our third theorem.

Theorem 3: Two distinct lines can not have more than one point in common.



Now let A be any point in plane π. By Axiom 1(a) there are two distinct lines m and n in π. If both m and n contain A then we have points B and C with B in n, C in m, $A \neq B$, and $A \neq C$ by Axiom 1(b). Furthermore, $B \neq C$ because of Theorem 3. Therefore, there is a line l containing B and C by Axiom 2. This line l can not contain A since if it did we would have A, B, and C all on l and $l = m = n$ which is impossible. Thus we have shown that in case $A \in m \cap n$ then there is a line l in π with A not on l. Clearly, if $A \notin n \cap m$ then either m or n does not contain A. Thus we have established the following theorem.

Theorem 4: If A is a point in a plane π then there is a line in π which does not contain A.

Exercises

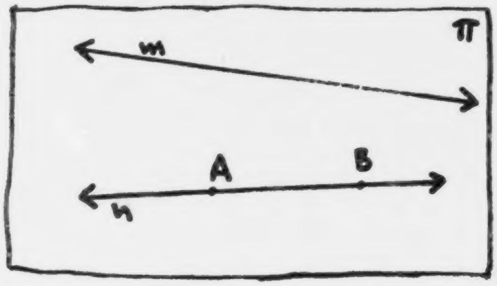
Prove the following theorems.

1. Theorem 5: If A is a point in plane π then there are at least two lines in π each containing point A.
2. Theorem 6: There are at least three non-concurrent lines in plane π.
3. Theorem 7: If each of two lines in π is parallel to the same line in π then they are parallel to each other.
4. Theorem 8: If m is any line in plane π then there are at least two points in π which are not in line m.
5. Theorem 9: If A is any point in plane π then there are at least two lines in π which do not contain A.

Contrived Passage CP7b

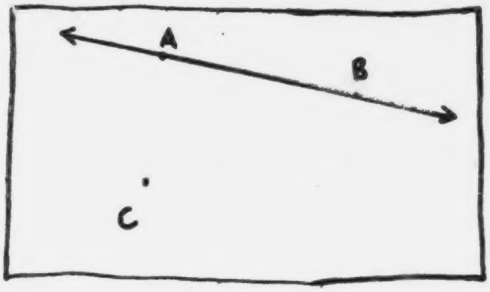
Some Logical Consequences of the Axioms

Theorem 1: If m is a line in plane π then there is a point in π which is not in m.



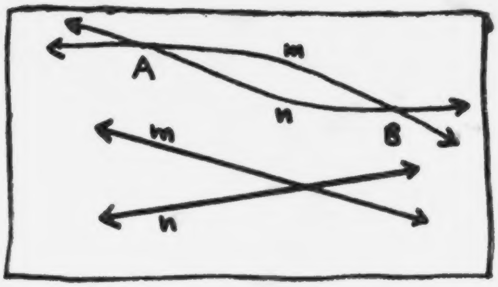
Suppose m is a line in plane π. By Axiom 1(a) there is a line n in π with $n \neq m$. By Axiom 1(b) there are distinct points A and B in line n. Now if both A and B were in m then we would have m = n which is not the case. Hence, at least one of the points A or B is not in m. Thus, we have the above theorem.

Theorem 2: There are at least three non-collinear points in a given plane.



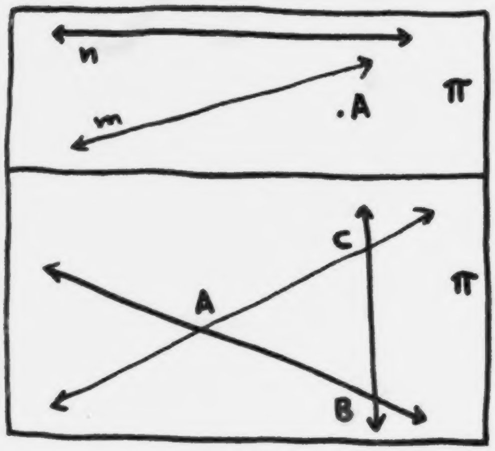
By Axiom 1(a) there exists a line m in a plane π. By Axiom 1(b) there are distinct points A and B on m. By Theorem 1 there exists a point C in plane π but not in line m. Hence, we have our second theorem.

Theorem 3: Two distinct lines can not have more than one point in common.



Let m and n be two distinct lines in a plane π. We know such lines exist by Axiom 1(a). Assume there are two distinct points A and B so that $A \in m \cap n$ and $B \in m \cap n$. Then m and n are two distinct lines each containing points A and B. But, this is impossible by Axiom 2. Hence, the above assumption is impossible and we have proven Theorem 3.

Theorem 4: If A is a point in a plane π then there is a line in π which does not contain A.



Let A be any point in π. By Axiom 1(a) there are two distinct lines m and n in π. If either m or n does not contain A then the theorem is true. Suppose m and n both contain A. Then there exist points B and C with B in m, C in n, $B \neq A$, and $C \neq A$ by Axiom 1(b). Furthermore, $B \neq C$ by Theorem 3. Now, there is a line l containing B and C by Axiom 2. This line l can not contain A since if it did we would have A, B, and C all on l and $l = m = n$ which is impossible. Thus, we have established the above theorem.

Exercoses

Prove the following theorems.

1. Theorem 5: If A is a point in plane π then there are at least two lines in π each containing point A.
2. Theorem 6: There are at least three non-concurrent lines in plane π.
3. Theorem 7: If each of two lines in π is parallel to the same line in π then they are parallel to each other.
4. Theorem 8: If m is any line in plane π then there are at least two points in π which are not in line m.
5. Theorem 9: If A is any point in plane π then there are at least two lines in π which do not contain A.

Contrived Passage CP9a

Cartesian Products and Relations

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Forming the set of all ordered pairs whose first component is from A and whose second component is from B we have $\{(1,a),(2,a),(3,a),(1,b),(2,b),(3,b)\}$. This set is denoted $\{1, 2, 3\} \times \{a,b\}$ or $A \times B$. $A \times B$ is read " A cross B" and is called the cartesian product of A and B. Notice that in this example an ordered pair $(x,y) \in \{1, 2, 3\} \times \{a, b\}$ if and only if $x \in \{1, 2, 3\}$ and $y \in \{a, b\}$.

Now letting $M = \{+, \Delta\}$ and $N = \{\Delta, *, \square\}$ we have,

$M \times N = \{(+,\Delta), (+,*), (+,\square), (\Delta,\Delta), (\Delta,*), (\Delta,\square)$. Notice that in this example $M \cap N = \{\Delta\} \neq \phi$. But, if the example above $A \times B = \phi$.

* * *

1. Give the roster name for each of the following cartesian products.
    a. $\{2, 3, 7\} \times \{0, 2, 3, 5, 6\}$
    b. $Z_3 \times Z_5$
    c. $\{x \mid x \in W \text{ and } x < 4\} \times \{z \mid z \in W \text{ and } z + 5 \leq 11\}$
    d. $A \times A$ where $A = \{x \mid x \in W, x \text{ is even, and } x < 9\}$

* * *

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$.

Let $R = \{(1,a), (2,b), (3,a)\}$.

Then $R \subset A \times B$. We say the set R is a relation from A to B. Suppose $S = \{(1,b), (3,b)\}$. Since $S \subset A \times B$, then S is also a relation from A to B.

* * *

2. Make up a relation from the first set to the second set in each part of exercise 1.
3. Make up a relation from the second set to the first set in each part of exercise 1.

* * *

Let $A = \{x \mid x \text{ is a male human being}\}$

Let $B = \{y \mid y \text{ is a female human being}\}$

Let $R = \{(a,b) \mid b \text{ is the daughter of } a \text{ and } a \text{ is the father of } b\}$

Then $R \subset A \times B$ and hence **R is a relation** from A to B. We might call this relation the father-daughter relation.

Let $U = \{x \mid x \text{ is a human being}\}$

Let $S = \{(a,b) \mid a \text{ is the brother of } b\}$

Then $S \subset U \times U$ and S is a relation from **U** to **U**. Notice that the set U is used as both the first and second set in the cartesian product in this example. In such a case we say S is a relation on U.

*   *   *

4. List two elements of the father-daughter relation.
5. List two elements of the brother relationship.
6. Which of the following sets are relations? Give the roster names of those which are relations.
   a. $\{2x + 3y \mid x \in W, y \in W, x < 5, \text{ and } y < 3\}$
   b. $Z_3 \times Z_4$
   c. $\{(x + 5, 3x) \mid x \in W \text{ and } x < 3\}$
   d. $\{1, 2, 6\} \times \{-1, -4, 3, 2\}$
   e. ϕ

*   *   *

Let W be the set of whole numbers.

Let $L = \{(x,y) \mid \exists\, c \in N \text{ such that } x + c = y\}$

Since $L \subset W \times W$ then L is a relation on W. L is such an important relation on W that we give it a special name. L is called the "less than" relation on W. We usually denote L by $<$ and write $2 < 3$ or $2 \; L \; 3$ to indicate that $(2,3) \in L$. Both of the statements $5 \; L \; 9$ and $(5,9) \in L$ mean the same thing.

* * *

7. a. Define the relation $>$ on W in terms of ordered pairs.

   b. List five ordered pairs in the relation $>$ on W.

   c. Rewrite $(a,b) \in >$ as $a > b$ for each of your answers to b.

8. Rewrite your answers to exercises 4 and 5 in the form $x \, R \, y$ where R represents the relation given in each problem.

* * *

Contrived Passage CP9b

Cartesian Products and Relations

Consider arbitary sets A and B and the set of all ordered pairs whose first component is an element of A and whose second component is an element of B. Such sets of ordered pairs are important in many branches of mathematics. We now make the following formal definition.

Definition 1: For all sets A and B,

$$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}.$$

$A \times B$ is read " A cross B " and is called the cartesian product of A and B.

An immediate consequence of this definition is the following theorem.

Theorem 1: $(a,b) \in A \times B \Leftrightarrow a \in A$ and $b \in B$.

Proof: $(\Rightarrow)$ Let $(a,b) \in A \times B$. Then $(a,b) \in \{(x,y) \mid x \in A \text{ and } y \in B\}$.

So, $a \in A$ and $b \in B$.

$(\Leftarrow)$ Let $a \in A$ and $b \in B$. Then $(a,b) \in \{(x,y) \mid x \in A \text{ and } y \in B\} = A \times B$.

Definition 2: Let A and B be arbitary sets.

Let $R \subset A \times B$.

Then R is called a relation from A to B.

This definition tells us that any subset of a cartesian product of two sets is a relation from the first set to the second set. When $R \subset A \times A$ we have a relation from A to A. In this case we say R is a relation on A. The idea of a relation is an important mathematical concept. Many relations can be found and are used in everyday life. Among these are all the biological relationships between people.

Let R be a relation. Since $R \subset A \times B$ for some sets A and B we know the elements of R must be ordered pairs. Thus we can write $(a,b) \in R$ to show the ordered pair (a,b) is in the relation R. Many times we write a R b to indicate that $(a,b) \in R$. Both of these statements mean exactly the same thing.

Exercises

1. Give the roster name for each of the following cartesian products.
   a. $\{2, 3, 7\} \times \{0, 2, 3, 5, 6\}$
   b. $Z_3 \times Z_5$
   c. $\{x \mid x \in W \text{ and } x < 4\} \times \{z \mid z \in W \text{ and } z + 5 \leq 11\}$
   d. $A \times A$ where $A = \{x \mid x \in W, x \text{ is even, and } x < 9\}$
2. Make up a relation from the first to the second set in each part of exercise 1.
3. Make up a relation from the second set to the first set in each part of exercise 1.
4. List two elements of the father-daughter relation.
5. List two elements of the brother relation.
6. Which of the following sets are relations? Give the roster names of those which are relations.
   a. $\{2x + 3y \mid x \in W, y \in W, x < 5, \text{ and } y \leq 3\}$
   b. $Z_3 \times Z_5$
   c. $\{(x + 5, 3x) \mid x \in W \text{ and } x \leq 3\}$
   d. $\{1, 2, 3\} \times \{-1, -4, 3, 2\}$
   e. ϕ
7. a. Define the relation "less than" on W in terms of ordered pairs.
   b. List five ordered pairs in the relation "less than" on W.
   c. Rewrite $(a,b) \in$ "less than" as $a < b$ for each of your answers to b.
8. Repeat exercise 7 for the relation "greater than" on W.
9. Rewrite your answers to exercises 4 and 5 in the form $x \, R \, y$ where R represents the relation given in each problem.

Contrived Passage CP10a

Mean Proportional in the Pythagorean Theorem

Since $ADC \leftrightarrow CDB$ in Figure I is a similarity, it follows from the definition of similarity that $\frac{AD}{CD} = \frac{DC}{DB}$. That is , it follows that $\frac{x}{h} = \frac{h}{y}$. So, $h^2 = x \cdot y$ and $h = \sqrt{x \cdot y}$.

The segment $\overline{CD}$ is said to be a mean proportional between the segments $\overline{AD}$ and $\overline{DB}$. You can compute the measure of $\overline{CD}$ by just computing the square root of the product of the measures of $\overline{AD}$ and $\overline{DB}$. So, we have the following theorem.

Theorem 1: The altitude to the hypotenuse of a right triangle is a mean proportional between the segments into which the foot of the altitude divides the hypotenuse.

Also in Figure I, since $ACD \leftrightarrow ABC$ is a similarity, it follows that $\frac{AD}{AC} = \frac{AC}{AB}$. So, $\overline{AC}$ is a mean proportional between $\overline{BD}$ and $\overline{BA}$. These results are useful in establishing one of the most important theorems in geometry, since from them it follows that

$$b^2 = x \cdot c \quad \text{and} \quad a^2 = y \cdot c \ .$$

Therefore, $a^2 + b^2 = y \cdot c + x \cdot c = (y + x) \cdot c = c^2$. Thus we have proved the Pythagorean Theorem.

Theorem 2: The sum of the squares of the measures of the legs of a right triangle is the square of the measure of the hypotenuse.

The Pythagorean Theorem is a surprising result, especially if you interpret it in terms of area. In Figure II, squares are drawn on each of the sides of a right triangle. Since the area-measure of a square is the square of the measure of oe of its sides, the Pythagorean Theorem tells us that the sum of the area-measures of the squares on the legs is the area-measure of the square on the hypotenuse.

It is possible to give other proofs of the Pythagorean Theorem in terms of areas. Considering figure III we can see how it is possible to disect the largest square into five regions, four of which are congruent to regions (1), (2), (3), and (4) and one of which is congruent to square (5).

Other area approaches to the Pythagorean Theorem are illustrated by Figure IV and Figure V. In Figure IV we have $(a + b)^2 = 4\,(\frac{1}{2}\, a \cdot b) + c^2$. So, $a^2 + 2\,ab + b^2 = 2\, a \cdot b + c^2$. Hence, $a^2 + b^2 = c^2$. In Figure V we have $c^2 = 4\,(\frac{1}{2}\, a \cdot b) + (b - a)^2 = 2\, a \cdot b + b^2 - 2\, a \cdot b + a^2 = a^2 + b^2$.

Finally it is possible to use the Pythagorean Theorem to find unknown lenghts in right triangles as illustrated by the following examples.

Example 1: Find BC in Figure VI.

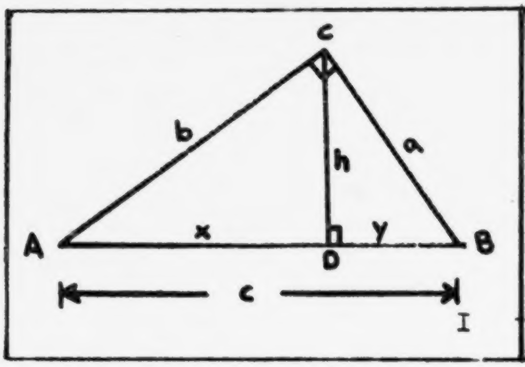
Since $a^2 + 7^2 = 25^2$, it follows that $a^2 + 49 = 625$ and that $a^2 = 576$. Hence, $a = \sqrt{576} = 24$. So, BC = 24.

Example 2: Find AB in Figure VII.

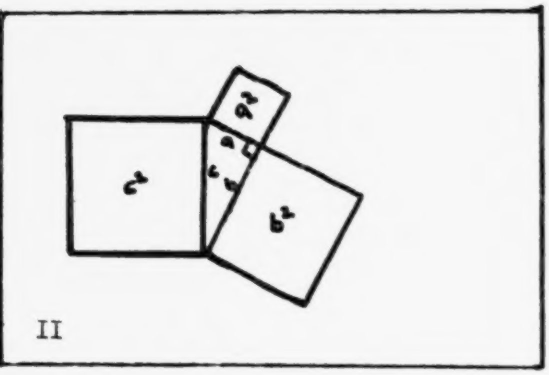
Since $c^2 = 6^2 + 10^2$, it follows that $c^2 = 136$ and that $c = \sqrt{136} = 2\sqrt{34}$. So, $AB = 2\sqrt{34}$.

Example 3: Find the measure of an altitude of the equilateral triangle in Figure VIII.

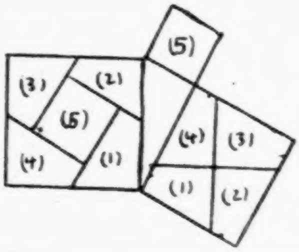
Since $(AB)^2 = (AD)^2 + n^2$, it follows that $n^2 = 6^2 - (\frac{6}{2})^2 = 36 - 9 = 27$. So, $h = \sqrt{27} = 3\sqrt{3}$.



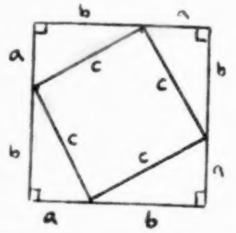
I



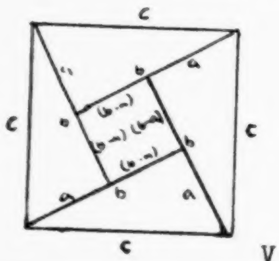
II


(5)
(3)
(2)
(5)
(4)
(1)
(4)
(3)
(1)
(2)

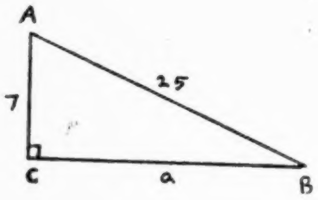
III


b
a
a
c
c
b
b
c
c
a
a
b

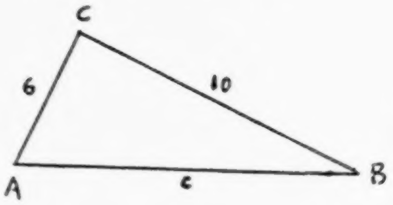
IV


c
a
b
a
c
c
b
(b-a)
b
a
b
a
c

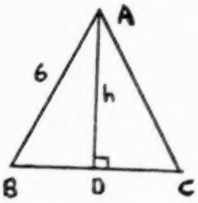
V



VI



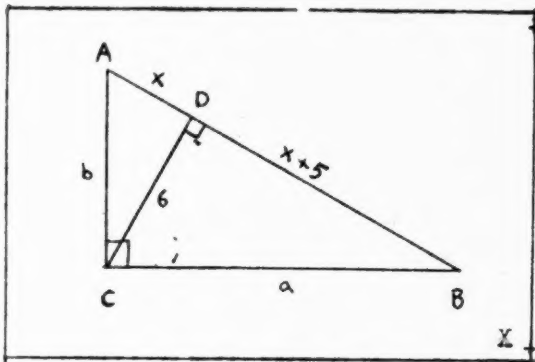
VII



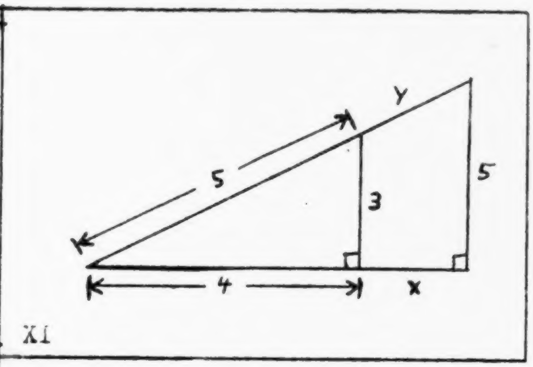
VIII

Exercises

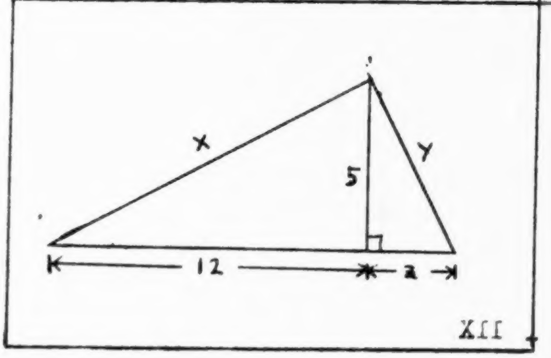
1. For right △ABC with hypotenuse c and legs a and b find the measure of the third side.
   a. $a = 2$, $b = 5$.
   b. $a = 9$, $b = 13$.
   c. $a = 3/4$, $c = 2$.
   d. $b = 2/3$, $c = 3/2$.
   e. $a = 3$, $b = \sqrt{a}$.
2. Find the indicated measures in Figure X.
3. Find the indicated measures in Figure XI.
4. Find the indicated measures in Figure XII.


A
x
D
b
x+5
6
C
a
B

X



XI



XII

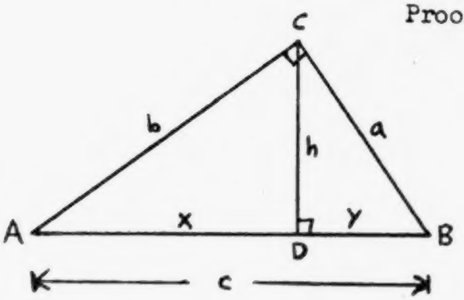
Contrived Passage CP10b

Mean Proportional and the Pythagorean Theorem

Definition 1: Given segments $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$.

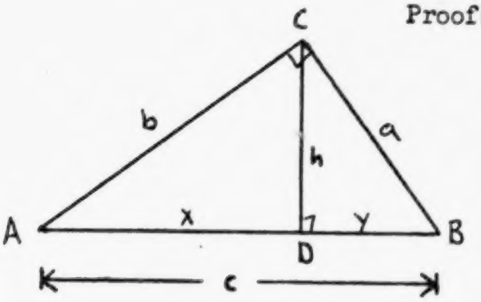
Then $\overline{CD}$ is the mean proportional between $\overline{AB}$ and $\overline{EF}$ if and only if $\frac{AB}{CD} = \frac{CD}{EF}$.

Theorem 1: The altitude to the hypotenuse of a right triangle is a **mean** proportional between the segments into which the foot of the altitude divides the hypotenuse.



Proof: Since $ADC \leftrightarrow CDB$ is a similarity it follows by definition of similarity that $\frac{AD}{CD} = \frac{DC}{DB}$ and the proof is complete.

Theorem 2: The sum of the squares of the measures of the legs of a right triangle is the square of the measure of the hypotenuse.



Proof: Since $ACD \leftrightarrow ABC$ is a similarity we have $\frac{AD}{AC} = \frac{AC}{AB}$ or $\frac{x}{b} = \frac{b}{c}$. So, $b^2 = x \cdot c$. Similarly, $CBD \leftrightarrow ABC$ is a similarity and $\frac{DB}{CB} = \frac{CB}{AB}$ or $\frac{y}{a} = \frac{a}{c}$. So, $a^2 = y \cdot c$. Thus, $a^2 + b^2 = x \cdot c + y \cdot c = (x + y) \cdot c = c^2$.

Theorem 2 is an important theorem in geometry. It is called the Pythagorean Theorem.

Exercises

1. For right $\triangle ABC$ with hypotenuse c and legs a and b find the measure of the third side.

   a. $a = 2, \ b = 5$.

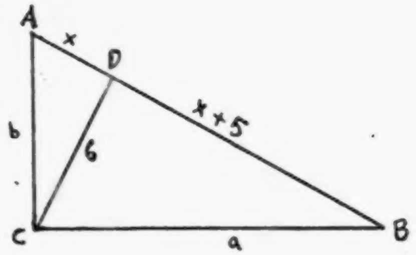
b. $a = 9$, $b = 13/2$.

c. $a = 3/4$, $c = 2$.

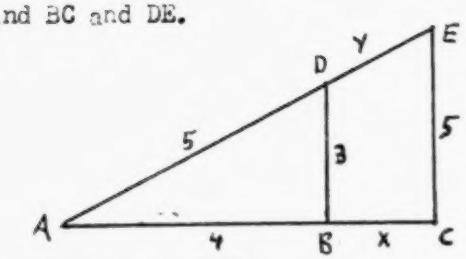
d. $b = 2/3$, $c = 3/2$.

e. $a = 3$, $b = \sqrt{a}$.

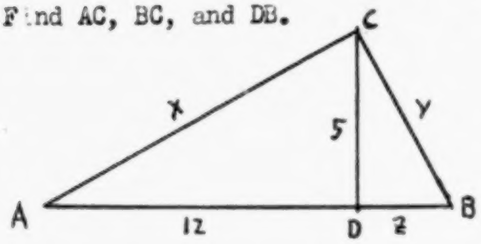
2. Find BC, AD, and BD.


A
x
D
x+5
b
6
C
a
B

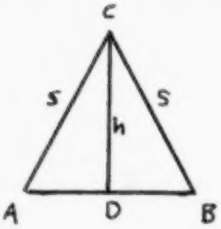
3. Find BC and DE.


E
D
Y
5
5
3
A
4
B
x
C

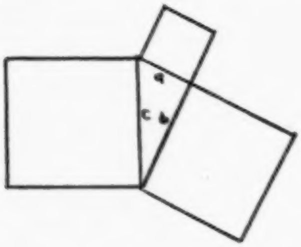
4. Find AC, BC, and DB.


C
x
5
Y
A
12
D
2
B

5. Find the altitude of an equalateral triangle in terms of the side length s.



6. Interpret the Pythagorean Theorem in terms of area.

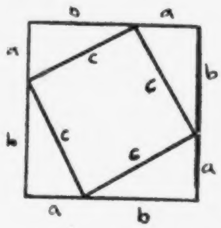


What are the areas of the squares?

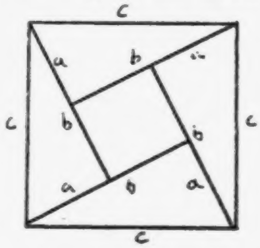
How are the areas of the squares related?

Why?

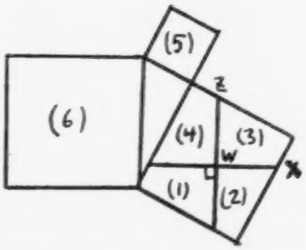
7. Use the figure to give an area proof of the Pythagorean Theorem.



8. Use the figure to give an area proof of the Pythagorean Theorem.



9. Let W be the point of intersection of the diagonals of the square. Let $\overline{XZ}$ be parallel to the hypotenuse of the right triangle. Let $\overline{WY} \perp \overline{XZ}$.



a. How can square (6) be disected into five regions four of which are congruent to (1), (2), (3), and (4) and one which is congruent to (5). (You can cut regions congruent to (1) through (5) out of some paper and try to fit them into region (6) if you have trouble.)

b. Explain how this proves the Pythagorean Theorem.

APPENDIX E

CMMT ANALYSIS COMPUTER PROGRAM

1. This program performs the CMMT analysis for mathematics text passages. Rater codings are punched on data cards. The output of this program includes the CMMT list, matrices, and proportions for each passage analyzed. Any number of passages may be analyzed in a single run.
2. Fortran Program Statements.

```
      DIMENSION A(10,7),B(10,10),C(7,7),M(45)
      N=0
   44 BLANK=0.0
      TOTAL=0.0
      DO 1 I=1,10
      DO 1 J=1,7
    1 A(I,J) = 0.0
      DO 2 I=1,10
      DO 2 J=1,10
    2 B(I,J)=0.0
      DO 3 I=1,7
      DO 3 J=1,7
    3 C(I,J)=0.0
      JJ1=0
      JJ2=0
      N=N+1
    9 READ(5,4) (M(K),K=1,45)
    4 FORMAT(15(I2,I1,I2))
      IF(M(1).EQ.99) GO TO 45
      IF(JJ1.EQ.0)WRITE(6,96) N
   96 FORMAT(19H0 LIST FOR PASSAGE ,I4)
      I=1
      J= 2
      K=3
    8 J1=M(I)
      J2=M(J)
      J3=M(K)
      IF(J1.EQ.88) GO TO 5
      IF(J1.GT.10.OR.J1.LT.0)   GO TO 46
      IF(J2.GT.7.OR.J2.LT.0)   GO TO 46
      IF(J3.LT.0) GO TO 46
      WRITE(6,60) J1,J2,J3
   60 FORMAT(5X,3I6)
      IF(J1.EQ.0) GO TO 6
      A(J1,J2)=A(J1,J2) + J3
      IF(JJ1.NE.0) B(JJ1,J1)=B(JJ1,J1)+1.0
      IF(JJ2.NE.0) C(JJ2,J2)=C(JJ2,J2)+1.0
      B(J1,J1)=B(J1,J1)+J3-1
      C(J2,J2)=C(J2,J2)+J3-1
      JJ1=J1
      JJ2=J2
      GO TO 7
    6 BLANK=BLANK+J3
    7 I=I+3
      J=J+3
      K=K+3
      TOTAL=TOTAL+J3
      IF(J.LT.45) GO TO 8
      IF(J.GT.45) GO TO 9
    5 CONTINUE
      WRITE(6,70) TOTAL
   70 FORMAT(14H0 TOTAL UNITS ,F6.0)
      PBLNK=BLANK/TOTAL
      WRITE (6,10) N,PBLNK
   10 FORMAT(31H0 PROPORTION OF BLANKS PASSAGE ,I4,F8.3)
      WRITE(6,50) N
   50 FORMAT(35H0 FREQUENCY MATRIX MODE 1  PASSAGE ,I3)
      DO 11 K=1,10
      WRITE(6,12)  (B(K,L),L=1,10)
   12 FORMAT(2H0 ,10F6.0)
   11 CONTINUE
      DO 13 I=1,10
      DO 13 J=1,10
   13 B(I,J)=B(I,J)/(TOTAL -BLANK-1.0)
      WRITE(6,51) N
   51 FORMAT(36H0 PROPORTION MATRIX MODE 1  PASSAGE ,I3)
      DO 14 K=1,10
   14 WRITE(6,15)  (B(K,L),L=1,10)
   15 FORMAT(2H0 ,10F6.3)
      WRITE(6,52) N
   52 FORMAT(43H0 PROPORTION EACH CATEGORY MODE 1  PASSAGE ,I3)
      DO 16 I=1,10
      B(I,1)=B(I,1)+B(I,2)+B(I,3)+B(I,4)+B(I,5)+B(I,6)
     C+B(I,7)+B(I,8)+B(I,9)+B(I,10)
      WRITE(6,17) B(I,1)
   17 FORMAT(3X,F6.3)
   16 CONTINUE
```

```
      P1=B(1,1)+B(2,1)+B(3,1)+B(4,1)
      P2=P1+B(6,1)
      P3=B(6,1)/P2
      P4=P1/P2
      P5=B(5,1)+B(6,1)+B(7,1)+B(8,1)+B(9,1)
      WRITE(6,61) P1
   61 FORMAT(27H0 PROPORTION OF EXPOSITION ,F6.3)
      WRITE(6,62) P2
   62 FORMAT(33H0 PROPORTION CONTENT DEVELOPMENT ,F6.3)
      WRITE(6,63) P3
   63 FORMAT(47H0 PROPORTION ACTIVITIES IN CONTENT DEVELOPMENT ,F6.3
      WRITE(6,64) P4
   64 FORMAT(47H0 PROPORTION EXPOSITION IN CONTENT DEVELOPMENT ,F6.3
      WRITE(6,65) P5
   65 FORMAT(39H0 PROPORTION REQUIRING OVERT RESPONSES ,F6.3)
      WRITE(6,53) N
   53 FORMAT(35H0 FREQUENCY MATRIX MODE 2  PASSAGE ,I3)
      DO 19 K=1,7
      WRITE(6,20) (C(K,L),L=1,7)
   20 FORMAT(2H0 ,7F6.0)
   19 CONTINUE
      DO 21 I=1,7
      DO 21 J=1,7
   21 C(I,J)=C(I,J)/(TOTAL-BLANK-1.0)
      WRITE(6,54) N
   54 FORMAT(36H0 PROPORTION MATRIX MODE 2  PASSAGE ,I3)
      DO 22 K=1,7
   22 WRITE(6,23) (C(K,L),L=1,7)
   23 FORMAT(2H0 ,7F6.3)
      WRITE(6,55) N
   55 FORMAT(43H0 PROPORTION EACH CATEGORY MODE 2  PASSAGE ,I3)
      DO 24 I=1,7
      C(I,1)=C(I,1)+C(I,2)+C(I,3)+C(I,4)+C(I,5)+C(I,6)+C(I,7)
      WRITE(6,25) C(I,1)
   25 FORMAT (3X,F6.3)
   24 CONTINUE

      P6=C(3,1)+C(4,1)+C(5,1)
      WRITE(6,26) P6
   26 FORMAT(31H0 PROPORTION MATH ILLUSTRATION ,F6.3)
      P7=P6+C(6,1)+C(7,1)
      WRITE(6,27) P7
   27 FORMAT(29H0 PROPORTION OF ILLUSTRATION ,F6.3)
      WRITE(6,59) N
   59 FORMAT(43H0 INTERACTION MATRIX MODES 1 AND 2 PASSAGE ,I3)
      DO 36 K=1,10
      WRITE(6,37) (A(K,L),L=1,7)
   37 FORMAT(2H0 ,7F6.0)
   36 CONTINUE
      GO TO 44
   46 WRITE(6,47) J1,J2,J3
   47 FORMAT(15H0 ERROR IN DATA,3I5)
   45 STOP
      END
```

3. Date Arrangement on Input Cards for a Given Passage.

First Message:
- Columns 1 and 2: Dimension 1 classification, right adjusted.
- Column 3: Dimension 2 classification.
- Columns 4 and 5: Weight of the message, right adjusted.

Second Message: Repeat above for columns 6 through 10.

.
.
.

Continue through the passage using 75 columns of each card. Columns 76 through 80 may be used for identification information. After the last message is punched, punch 88 in the following two columns. Each passage to be analyzed starts on a new card.

4. Data Deck Order.

First Passage Cards.

Second Passage Cards.

.
.
.

Last Passage Cards.

Final Card of Data Deck: Punch 99 in columns 1 and 2.

APPENDIX F

COMPUTER PROGRAMS FOR SCOTT RELIABILITY COEFFICIENTS

1. These programs compute the Scott reliability coefficients for each dimension of the CMMT category system. The programs compute the coefficients for any number of pairs of subjects.
2. Fortran Statements of the Dimension 1 Program.

```
      DIMENSION I1(38),I2(38),P(10)
      NN=0
   20 NN=NN+1
      N=0
   21 T=0.0
      A=0.0
      DO 1 K=1,10
    1 P(K)=0.0
   10 READ(5,2) (I1(K),K=1,38)
    2 FORMAT(38I2)
      IF(I1(1).EQ.-1) GO TO 6
      IF(I1(1).EQ.99) GO TO 20
      READ(5,2) (I2(K),K=1,38)
      DO 3 K=1,38
      IF(I1(K).EQ.0) GO TO 33
      IF(I1(K).EQ.I2(K)) A=A+1.0
      M=I1(K)
      P(M)=P(M)+1.0
      M=I2(K)
      P(M)=P(M)+1.0
      T=T+1.0
    3 CONTINUE
      GO TO 10
   33 PO=A/T
      DO 4 K=1,10
    4 P(K)=(P(K)/(2*T))**2
      PE=P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7)+P(8)+P(9)+P(10)
      R=(PO-PE)/(1-PE)
      N=N+1
      WRITE(6,7) NN,N
    7 FORMAT(20H0DIMENSION 1  PAIR (,I4,2H ,,I4,2H ))
      WRITE(6,8) PO
    8 FORMAT(25H0PROPORTION OF AGREEMENT ,F8.6)
      WRITE (6,9) R
    9 FORMAT(22H0RELIABILITY ESTIMATE ,F8.6)
      GO TO 21
    6 STOP
      END
```

3. Fortran Statements of the Dimension 2 Program.

```
    DIMENSION I1(38),I2(38),P(7)
    NN=0
20  NN=NN+1
    N=0
21  T=0.0
    A=0.0
    DO 1 K=1,7
 1  P(K)=0.0
10  READ(5,2) (I1(K),K=1,38)
 2  FORMAT(38I2)
    IF(I1(1).EQ.-1) GO TO 6
    IF(I1(1).EQ.99) GO TO 20
    READ(5,2) (I2(K),K=1,38)
    DO 3 K=1,38
    IF(I1(K).EQ.0) GO TO 33
    IF(I1(K).EQ.I2(K)) A=A+1.0
    M=I1(K)
    P(M)=P(M)+1.0
    M=I2(K)
    P(M)=P(M)+1.0
    T=T+1.0
 3  CONTINUE
    GO TO 10
33  PO=A/T
    DO 4 K=1,7
 4  P(K)=(P(K)/(2*T))**2
    PE=P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7)
    R=(PO-PE)/(1-PE)
    N=N+1
    WRITE(6,7) NN,N
 7  FORMAT(20H0DIMENSION 2  PAIR (,I4,2H ,,I4,2H ))
    WRITE(6,8) PO
 8  FORMAT(25H0PROPORTION OF AGREEMENT ,F8.6)
    WRITE (6,9) R
 9  FORMAT(22H0RELIABILITY ESTIMATE ,F8.6)
    GO TO 21
 6  STOP
    END
```

4. Data Arrangement for Input Cards for a Given Subject on Dimension n, n = 1,2.

First Card:

Columns 1 and 2: Subject's dimension n rating of first message, right adjusted.

Columns 3 and 4: Subject's dimension n rating of second message, right adjusted.

.
.
.

Repeat for as many columns as necessary up through column 76.

Columns 77 through 80: Identification information.

Second Card (If necessary) Continue above.

.
.
.

Continue until all messages are punched.

Note: Messages coded 0 are not punched on the data cards. Each pair of columns before the end of the passage must have a positive entry.

5. Data Deck Order.

- a. Cards for subject pair (m,n):
    - First card for subject m.
    - First card for subject n.
    - .
    - .
    - .
    - Last card for subject m.
    - Last card for subject n.
- b. Order of pairs in deck:
    - First Subgroup:
        - First subject pair cards.
        - Second subject pair cards.
        - .
        - .
        - .
        - Last subject pair cards.
        - End of subgroup card: Punch 99 in columns 1 and 2.
    - .
    - .
    - .
    - Repeat for as many subgroups as desired.
    - .
    - .
    - .
    - Last data card in deck: Punch -1 in columns 1 and 2.

APPENDIX G

COMPUTER PROGRAMS FOR SCORING CRITERION PASSAGES

1. These programs compute between- and within-rater scores for subjects using the CMMT technique to code a given passage. The output consists of scores for each dimension, overall scores, and the KR-20 reliability estimate for each CMMT dimension. The programs will score up to 60 subjects' ratings on a given passage.

2. a. Fortran Program Statements for One Card Passages and Between-Rater Scores.

```
101 DIMENSION IC1(38),IC2(38),T1(10),T2(7)
102 DIMENSION IS1(38),IS2(38),P1(10),P2(7)
103 DIMENSION R1(60),R2(60),PV1(38),PV2(38)
201 NN=0
202 DO 1 K=1,10
  1 T1(K)=0.0
204 DO 2 K=1,7
  2 T2(K)=0.0
    DO 42 K=1,60
    R1(K)=0.0
 42 R2(K)=0.0
104 DO 34 K=1,38
    PV1(K)=0.0
 34 PV2(K)=0.0
105 READ(5,3) (IC1(K),K=1,38)
106 READ(5,3) (IC2(K),K=1,38)
  3 FORMAT(38I2)
107 DO 4 K=1,38
    IF(IC1(K).EQ.0) IT=K-1
    IF(IC1(K).EQ.0) GO TO 19
    M1=IC1(K)
    M2=IC2(K)
    T1(M1)=T1(M1)+1.0
  4 T2(M2)=T2(M2)+1.0
 19 X=T2(1)+T2(2)+T2(3)+T2(4)+T2(5)+T2(6)+T2(7)
 17 READ(5,3) (IS1(K),K=1,38)
203 IF(IS1(1).EQ.-1)GO TO 18
108 READ(5,3) (IS2(K),K=1,38)
    DO 5 K=1,10
  5 P1(K)=0.0
    DO 6 K=1,7
    6 P2(K)=0.0
  109 DO 7 K=1,38
      IF(IS1(K).EQ.0) GO TO 20
      M=IC1(K)
      N=IS1(K)
      IF(M.EQ.N) P1(M)=P1(M)+1.0
      IF(M.EQ.N) PV1(K)=PV1(K)+1.0
      M=IC2(K)
      N=IS2(K)
      IF(M.EQ.N) PV2(K)=PV2(K)+1.0
    7 IF(M.EQ.N) P2(M)=P2(M)+1.0
   20 S1=P1(1)+P1(2)+P1(3)+P1(4)+P1(5)+P1(6)+P1(7)+P1(8)+P1(9)+P1(10
      S2=P2(1)+P2(2)+P2(3)+P2(4)+P2(5)+P2(6)+P2(7)
      NN=NN+1
      WRITE (6,8) NN
    8 FORMAT(30H0BETWEEN RATER SCORES SUBJECT ,I6)
      WRITE(6,9) X,S1,S2
    9 FORMAT(42H0NUMBER OF ITEMS AND NUMBER RIGHT EACH DIM,3F8.2)
      R1(NN)=S1
      R2(NN)=S2
      DO 10 K=1,10
      IF(T1(K).NE.0.0) P1(K)=P1(K)/T1(K)
   10 IF(T1(K).EQ.0.0) P1(K)=-1
      WRITE(6,11) (P1(K),K=1,10)
   11 FORMAT(26H0SCORE EACH CATEGORY DIM 1,10F6.3)
      S1=S1/X
      WRITE(6,12) S1
   12 FORMAT(13H0 SCORE DIM 1,F8.3)
      DO 13 K=1,7
      IF(T2(K).NE. 0.0) P2(K)=P2(K)/T2(K)
   13 IF(T2(K).EQ. 0.0) P2(K)=-1
      WRITE(6,14)(P2(K),K=1,7)
   14 FORMAT(26H0SCORE EACH CATEGORY DIM 2,7F6.3)
      S2=S2/X
      WRITE(6,15) S2
   15 FORMAT(13H0 SCORE DIM 2,F8.3)
      S1=(S1+S2)/2.0
      WRITE(6,16) S1
   16 FORMAT(15H0 OVERALL SCORE,F8.3)
      GO TO 17
   18 SR01=0.0
      SR02=0.0
      DO 32 K=1,IT
      PV1(K)=PV1(K)/NN
   32 PV2(K)=PV2(K)/NN
      DO 33 K=1,IT
    SPQ1=SPQ1+PV1(K)*(1-PV1(K))
 33 SPQ2=SPQ2+PV2(K)*(1-PV2(K))
    XMI1=0.0
    XMI2=0.0
    DO 30 K=1,NN
    XMI1=XMI1+R1(K)
 30 XMI2=XMI2+R2(K)
    XMI1=XMI1/NN
    XMI2=XMI2/NN
    VY1=0.0
    VY2=0.0
    DO 36 K=1,NN
    R1(K)=R1(K)-XMI1
 36 R2(K)=R2(K)-XMI2
    DO 31 K=1,NN
    VY1=VY1+R1(K)**2
 31 VY2=VY2+R2(K)**2
    VY1=VY1/NN
    VY2=VY2/NN
    Z=IT
    RKK1=(Z/(Z-1))*(1-(SPQ1/VY1))
    RKK2=(Z/(Z-1))*(1-(SPQ2/VY2))
    WRITE(6,35)(R1(K),K=1,60)
 35 FORMAT(23H0DEVIATION SCORES DIM 1,6(/5X,10F6.2))
    WRITE (6,40)(R2(K),K=1,60)
 40 FORMAT(23H0DEVIATION SCORES DIM 2,6(/5X,10F6.2))
    WRITE(6,37) VY1
 37 FORMAT(16H0VARIANCE DIM 1 ,F8.3)
    WRITE(6,38) VY2
 38 FORMAT(16H0VARIANCE DIM 2 ,F8.3)
    WRITE (6,39) RKK1
 39 FORMAT(12H0KR20 DIM 1 ,F6.4)
    WRITE(6,41) RKK2
 41 FORMAT(12H0KR20 DIM 2 ,F6.4)
    STOP
    END
```

b. Modifications for Two Card Passages.

Replace 38 with 76 in program statements numbered 101 through 109 and 17.

c. Modifications for Within-Rater Scores.

Remove statements numbered 17, 203 and 108 from the positions given above and place them between statements numbered 34 and 105 so that the order of statements is 34, 17, 203, 108, 105.

Remove statements numbered 202, 1, 204, and 2, and place them between statements numbered 3 and 107 so that the order is 3, 202, 1, 204, 2, 107.

3. Data Arrangement for Input Cards for a Given Subject (or Criterion Scores) on Dimension n, n = 1,2.

First Card:

Columns 1 and 2: Subject's dimension n rating of first message, right adjusted.

Columns 3 and 4: Subject's dimension n rating on second message, right adjusted.

.
.
.

Repeat for as many columns as necessary up through column 76.

Column 77 through 80: Identification information.

Second Card (If necessary): Continue above.

Note: Messages rated 0 are not entered on the data cards. Each pair of columns before the end of the passage must have a positive entry.

4. Data Deck Order.

- a. Between-Rater Scores:
  - Dimension 1 criterion card(s).
  - Dimension 2 criterion card(s).
  - Subject 1 dimension 1 card(s).
  - Subject 1 dimension 2 card(s).
  - .
  - .
  - .
  - Subject n dimension 1 card(s), $n \leq 60$.
  - Subject n dimension 2 card(s), $n \leq 60$.
  - Final card in deck: Punch -1 in columns 1 and 2.
- b. Within-Rater Scores:
  - Subject 1 dimension 1 card(s), first rating.
  - Subject 1 dimension 2 card(s), first rating.
  - Subject 1 dimension 1 card(s), second rating.
  - Subject 1 dimension 2 card(s), second rating.
  - .
  - .
  - .
  - Repeat for each subject up to n, $n \leq 60$.
  - Final card in deck: Punch -1 in columns 1 and 2.

APPENDIX H

COMPUTER PROGRAMS FOR DETERMINING CRITERION RATINGS

1. These programs compute the most frequent ratings of messages on both CMMT dimensions for any number of raters on any number of passages.
2. a. Fortran Program Statements for One Card Passages.

```
   DIMENSION IA(38),IB(38),IC(10,38),ID(7,38)
20 DO 1 J=1,38
   DO 2 K=1,10
 2 IC(K,J) = 0
   DO 3 L=1,7
 3 ID(L,J) = 0
 1 CONTINUE
 6 READ (5,4) (IA(K), K=1,38)
 4 FORMAT (38I2)
   IF(IA(1).EQ.-1) GO TO 7
   IF(IA(1).EQ.-2) GO TO 21
   READ(5,4) (IB(K),K=1,38)
   DO 5 J=1,38
   IF(IA(J).EQ.0) GO TO 6
   K = IA(J)
   L = IB(J)
   IC(K,J) = IC(K,J) + 1
 5 ID(L,J) = ID(L,J) + 1
   GO TO 6
 7 DO 8 K = 1,38
 8 WRITE(6,9) (IC(J,K), J=1,10)
 9 FORMAT (1H ,10I5)
   DO 10 K=1,38
11 FORMAT (1H ,7I5)
10 WRITE(6,11) (ID(J,K),J=1,7)
   GO TO 20
21 STOP
   END
```

b. Fortran Program Statements of Two Card Passages.

```
      DIMENSION IA(52),IB(52),IC(10,52),ID(7,52)
   20 DO 1 J=1,52
      DO 2 K=1,10
    2 IC(K,J) = 0
      DO 3 L=1,7
    3 ID(L,J) = 0
    1 CONTINUE
    6 READ (5,4) (IA(K), K=1,52)
    4 FORMAT (36I2/16I2)
      IF(IA(1).EQ.-1) GO TO 7
      IF(IA(1).EQ.-2) GO TO 21
      READ(5,4) (IB(K),K=1,52)
      DO 5 J=1,52
      IF(IA(J).EQ.0) GO TO 6
      K = IA(J)
      L = IB(J)
      IC(K,J) = IC(K,J) + 1
    5 ID(L,J) = ID(L,J) + 1
      GO TO 6
    7 DO 8 K = 1,52
    8 WRITE(6,9) (IC(J,K), J=1,10)
    9 FORMAT (1H ,10I5)
      DO 10 K=1,52
   11 FORMAT (1H ,7I5)
   10 WRITE(6,11) (ID(J,K),J=1,7)
      GO TO 20
   21 STOP
      END
```

3. Data Arrangement for Input Cards for a Given Subject on Dimension n, n = 1, 2.

   Same as item 3 of Appendix G.

4. Data Deck Order.

   Subject 1 dimension 1 card(s).

   Subject 1 dimension 2 card(s).

   .
   .
   .

Subject n dimension 1 card(s).

Subject n dimension 2 card(s).

End card(s) for first passage: Punch -1 in columns 1 and 2. Use two of these for two card passages.

.
.
.

Repeat for as many passages as desired.

Last card(s) in deck: Punch -2 in columns 1 and 2. Use two of these for two card passages.